Extending n-sequences

Christian Hokaj and Kendra Plante

April 2, 2024

Classical Multiplier Sequences

Definition

We call a polynomial *hyperbolic* if it has only real zeroes.

Classical Multiplier Sequences

Definition

We call a polynomial *hyperbolic* if it has only real zeroes.

Definition

Let *CMS* denote the set of real sequences $\{\gamma_k\}_{k=0}^{\infty}$ with the following property: For any $n \in \mathbb{N}$, $\sum_{k=0}^{n} \gamma_k a_k x^k$ is hyperbolic whenever $\sum_{k=0}^{n} a_k x^k$ is hyperbolic.

Classical Multiplier Sequences

Definition

We call a polynomial *hyperbolic* if it has only real zeroes.

Definition

Let CMS denote the set of real sequences $\{\gamma_k\}_{k=0}^{\infty}$ with the following property: For any $n \in \mathbb{N}$, $\sum_{k=0}^{n} \gamma_k a_k x^k$ is hyperbolic whenever $\sum_{k=0}^{n} a_k x^k$ is hyperbolic.

Definition

Let CMS_n denote the set of real sequences of $n+1$ terms $\{\gamma_k\}_{k=0}^{n}$ with the following property: For any $m \leq n$, $\sum_{k=0}^{m} \gamma_k a_k x^k$ is hyperbolic whenever $\sum_{k=0}^{m} a_k x^k$ is hyperbolic.

Classical Multiplier Sequences

Proposition (Pólya and Schur)

A sequence of the form $\{\dots, \gamma_n, 0, \gamma_{n+2}, \dots\}$, *where* $\gamma_n \neq 0$ *and* $\gamma_{n+2} \neq 0$ *is not a classical multiplier sequence.*

Classical Multiplier Sequences

Proposition (Pólya and Schur)

A sequence of the form $\{\dots, \gamma_n, 0, \gamma_{n+2}, \dots\}$, *where* $\gamma_n \neq 0$ *and* $\gamma_{n+2} \neq 0$ *is not a classical multiplier sequence.*

Corollary

If $\{\gamma_k\}_{k=0}^{n}$ *is an n-sequence and* $\gamma_n = 0$, *then* $\{\gamma_k\}_{k=0}^{n} \cup \{a\}$ *for* $a \neq 0$ *is not an* $(n+1)$*-sequence.*

Classical Multiplier Sequences

Proposition (Pólya and Schur)

A sequence of the form $\{\ldots, \gamma_n, 0, \gamma_{n+2}, \ldots\}$, *where* $\gamma_n \neq 0$ *and* $\gamma_{n+2} \neq 0$ *is not a classical multiplier sequence.*

Corollary

If $\{\gamma_k\}_{k=0}^n$ *is an* n*-sequence and* $\gamma_n = 0$, *then* $\{\gamma_k\}_{k=0}^n \cup \{a\}$ *for* $a \neq 0$ *is not an* $(n+1)$*-sequence.*

Proposition (Pólya and Schur)

A polynomial of the form $f(x) = \sum_{k=0}^m a_k x^k + \sum_{k=m+3}^n a_k x^k$ *is not hyperbolic.*

Algebraic Characterization

Definition

Let $\mathcal{H}_n$ denote the set of polynomials over $\mathbb{R}$ of degree at most n whose roots are all real and of the same sign.

Algebraic Characterization

Definition

Let $\mathcal{H}_n$ denote the set of polynomials over $\mathbb{R}$ of degree at most n whose roots are all real and of the same sign.

Theorem (Pólya-Schur)

If $\{\gamma_k\}_{k=0}^{\infty} \in \mathbb{R}^{\omega}$, *then* $\{\gamma_k\}_{k=0}^{\infty} \in CMS$ *if and only if*

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \in \mathcal{H}_n$$

for all $n \in \mathbb{N}$.

Algebraic Characterization

Corollary

If $\{\gamma_k\}_{k=0}^n \in \mathbb{R}^{n+1}$, *then* $\{\gamma_k\}_{k=0}^n \in CMS_n$ *if and only if*

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \in \mathcal{H}_n$$

Algebraic Characterization

Corollary

If $\{\gamma_k\}_{k=0}^n \in \mathbb{R}^{n+1}$, *then* $\{\gamma_k\}_{k=0}^n \in CMS_n$ *if and only if*

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \in \mathcal{H}_n$$

We call g_n the n-th Jensen polynomial associated to $\{\gamma_k\}_{k=0}^n$ and write $g_n \sim \{\gamma_k\}_{k=0}^n$.

Topological Equivalence

Proposition

Let

$$\phi\left(\sum_{k=0}^{n}\binom{n}{k}\gamma_k x^k\right) = \{\gamma_k\}_{k=0}^{n}$$

Then $\phi : \mathcal{H}_n \to CMS_n$ *is a homeomorphism for each* $n \in \mathbb{N}$.

Topological Equivalence

Proposition

Let

$$\phi\left(\sum_{k=0}^{n}\binom{n}{k}\gamma_k x^k\right) = \{\gamma_k\}_{k=0}^{n}$$

Then $\phi : \mathcal{H}_n \to CMS_n$ *is a homeomorphism for each* $n \in \mathbb{N}$.

Remark

For each $n \in \mathbb{N}$, *we interpret* $\mathcal{H}_n$ *as a topological subspace of* $\mathbb{R}_n[x]$ *with the compact convergence topology. Similarly, we interpret* CMS_n *as a topological subspace of* $\mathbb{R}^{n+1}$ *with the Euclidean metric.*

Main Question

How is CMS related to each CMS_n? How is CMS_n related to CMS_m for $n \neq m$?

Main Question

How is CMS related to each CMS_n? How is CMS_n related to CMS_m for $n \neq m$?

Definition

Let $\{\gamma_0, \gamma_1, \ldots, \gamma_n\} \in CMS_n$ and $\{\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots, \gamma_m\} \in CMS_m$ for $m > n$. Then we say that $\{\gamma_k\}_{k=0}^{m}$ is an *extension* of $\{\gamma_k\}_{k=0}^{n}$.

Main Question

How is CMS related to each CMS_n? How is CMS_n related to CMS_m for $n \neq m$?

Definition

Let $\{\gamma_0, \gamma_1, \ldots, \gamma_n\} \in CMS_n$ and $\{\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots, \gamma_m\} \in CMS_m$ for $m > n$. Then we say that $\{\gamma_k\}_{k=0}^{m}$ is an *extension* of $\{\gamma_k\}_{k=0}^{n}$.

Note that, for any m-sequence $\{\gamma_k\}_{k=0}^{m}$, and any $n < m$, $\{\gamma_k\}_{k=0}^{n}$ is an n-sequence.

Main Question

How is CMS related to each CMS_n? How is CMS_n related to CMS_m for $n \neq m$?

Definition

Let $\{\gamma_0, \gamma_1, \ldots, \gamma_n\} \in CMS_n$ and $\{\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots, \gamma_m\} \in CMS_m$ for $m > n$. Then we say that $\{\gamma_k\}_{k=0}^{m}$ is an *extension* of $\{\gamma_k\}_{k=0}^{n}$.

Note that, for any m-sequence $\{\gamma_k\}_{k=0}^{m}$, and any $n < m$, $\{\gamma_k\}_{k=0}^{n}$ is an n-sequence.

Question

For any $m > n$, can every n-sequence be extended to an m-sequence? If not, can we characterize which ones can be extended, and how?

Known Results

Theorem (Craven and Csordas)

The following hold:

Known Results

Theorem (Craven and Csordas)

The following hold:

- *For each integer $n \geq 3$, there exists an n-sequence which cannot be extended to an $(n+1)$-sequence.*

Known Results

Theorem (Craven and Csordas)

The following hold:

- *For each integer $n \geq 3$, there exists an n-sequence which cannot be extended to an $(n+1)$-sequence.*
- *If $\{\gamma_k\}_{k=0}^{n}$ is a sequence of real numbers with $\gamma_0 = 1$ and $\gamma_{k-1}^2 \geq 4(1 - 1/k)\gamma_k\gamma_{k-2}$ for $k \in \{2, 3, \ldots, n\}$, then $\{\gamma_k\}_{k=0}^{n}$ is an n-sequence which is extendable to CMS.*

Known Results

Theorem (Craven and Csordas)

The following hold:

- *For each integer $n \geq 3$, there exists an n-sequence which cannot be extended to an $(n+1)$-sequence.*
- *If $\{\gamma_k\}_{k=0}^n$ is a sequence of real numbers with $\gamma_0 = 1$ and $\gamma_{k-1}^2 \geq 4(1-1/k)\gamma_k\gamma_{k-2}$ for $k \in \{2, 3, \ldots, n\}$, then $\{\gamma_k\}_{k=0}^n$ is an n-sequence which is extendable to CMS.*
- *If g_n is a Jensen polynomial of degree n with two consecutive non-simple zeros, then the associated n-sequence cannot be extended to an $(n+1)$-sequence.*

Characterization of Boundary and Interior

Theorem (H-P)

For each $n \in \mathbb{N}_{n\geq 2}$, $p \in \mathcal{H}_n$ is a boundary point if and only if $p(0) = 0$ or p has a zero of multiplicity $m \geq 2$.

Characterization of Boundary and Interior

Theorem (H-P)

For each $n \in \mathbb{N}_{n\geq 2}$, $p \in \mathcal{H}_n$ is a boundary point if and only if $p(0) = 0$ or p has a zero of multiplicity $m \geq 2$.

Corollary

$\mathcal{H}_n$ has nonempty interior in $\mathbb{R}_n[x]$ for all $n \in \mathbb{N}$. Equivalently, CMS_n has nonempty interior in $\mathbb{R}^{n+1}$ for all $n \in \mathbb{N}$.

Characterization of Boundary and Interior

Theorem (H-P)

For each $n \in \mathbb{N}_{n\geq 2}$, $p \in \mathcal{H}_n$ is a boundary point if and only if $p(0) = 0$ or p has a zero of multiplicity $m \geq 2$.

Corollary

$\mathcal{H}_n$ has nonempty interior in $\mathbb{R}_n[x]$ for all $n \in \mathbb{N}$. Equivalently, CMS_n has nonempty interior in $\mathbb{R}^{n+1}$ for all $n \in \mathbb{N}$.

Example

- $(x+1)^n \sim \{1, 1, \dots\}$ is a boundary point of $\mathcal{H}_n$ and CMS_n.
- $x + 1 \sim \{1, \frac{1}{n}, 0, 0, \dots\}$ is an interior point of $\mathcal{H}_n$ and CMS_n.

Constructing n-sequences without Extensions

Consider $(x+1)^n \in \mathcal{H}_{n+1}$, $n \geq 2$.

Constructing n-sequences without Extensions

Consider $(x+1)^n \in \mathcal{H}_{n+1}$, $n \geq 2$.

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{n+1-k}{n+1} x^k \sim \left\{ \frac{n+1-k}{n+1} \right\}_{k=0}^{n+1}$$

Constructing n-sequences without Extensions

Consider $(x+1)^n \in \mathcal{H}_{n+1}$, $n \geq 2$.

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{n+1-k}{n+1} x^k \sim \left\{ \frac{n+1-k}{n+1} \right\}_{k=0}^{n+1}$$

- Thus, $\left\{ \frac{n+1-k}{n+1} \right\}_{k=0}^{n+1} \in CMS_{n+1}$

Constructing n-sequences without Extensions

Consider $(x+1)^n \in \mathcal{H}_{n+1}$, $n \geq 2$.

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{n+1-k}{n+1} x^k \sim \left\{ \frac{n+1-k}{n+1} \right\}_{k=0}^{n+1}$$

- Thus, $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \in CMS_{n+1}$
- Is $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \in CMS_{n+2}$?

Constructing n-sequences without Extensions

Is $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \in CMS_{n+2}$?

Constructing n-sequences without Extensions

Is $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \in CMS_{n+2}$? We compute the $(n+2)$th Jensen polynomial:

Constructing n-sequences without Extensions

Is $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \in CMS_{n+2}$? We compute the $(n+2)$th Jensen polynomial:

$$g(x) = \sum_{k=0}^{n} \binom{n+2}{k} \frac{n+1-k}{n+1} x^k$$

Constructing n-sequences without Extensions

Is $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \in CMS_{n+2}$? We compute the $(n+2)$th Jensen polynomial:

$$g(x) = \sum_{k=0}^{n} \binom{n+2}{k} \frac{n+1-k}{n+1} x^k = \frac{n+2}{x^{n+2}} \int_0^x t(t+1)^n dt$$

Constructing n-sequences without Extensions

Is $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \in CMS_{n+2}$? We compute the $(n+2)$th Jensen polynomial:

$$g(x) = \sum_{k=0}^{n} \binom{n+2}{k} \frac{n+1-k}{n+1} x^k = \frac{n+2}{x^{n+2}} \int_0^x t(t+1)^n dt$$

- If n is odd, g has no real zeroes. If n is even, then g has exactly one real zero.

Constructing n-sequences without Extensions

Is $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \in CMS_{n+2}$? We compute the $(n+2)$th Jensen polynomial:

$$g(x) = \sum_{k=0}^{n} \binom{n+2}{k} \frac{n+1-k}{n+1} x^k = \frac{n+2}{x^{n+2}} \int_0^x t(t+1)^n dt$$

- If n is odd, g has no real zeroes. If n is even, then g has exactly one real zero.
- Therefore, $g \notin \mathcal{H}_{n+2}$, which implies $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \notin CMS_{n+2}$

Constructing n-sequences without Extensions

Is $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \in CMS_{n+2}$? We compute the $(n+2)$th Jensen polynomial:

$$g(x) = \sum_{k=0}^{n} \binom{n+2}{k} \frac{n+1-k}{n+1} x^k = \frac{n+2}{x^{n+2}} \int_0^x t(t+1)^n dt$$

- If n is odd, g has no real zeroes. If n is even, then g has exactly one real zero.
- Therefore, $g \notin \mathcal{H}_{n+2}$, which implies $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1} \cup \{0\} \notin CMS_{n+2}$
- Thus, $\left\{\frac{n+1-k}{n+1}\right\}_{k=0}^{n+1}$ cannot be extended.

Further Results

- The n-sequence associated with x^m for any $m \leq n$ can always be extended to CMS.

Further Results

- The n-sequence associated with x^m for any $m \leq n$ can always be extended to CMS.
- There are some interior points which can be extended, e.g. $(x+1)(x+2) \in \mathcal{H}_3$, and some which cannot, e.g. $(x+1)(x+2)(x+3) \in \mathcal{H}_4$.

Further Results

- The n-sequence associated with x^m for any $m \leq n$ can always be extended to CMS.
- There are some interior points which can be extended, e.g. $(x+1)(x+2) \in \mathcal{H}_3$, and some which cannot, e.g. $(x+1)(x+2)(x+3) \in \mathcal{H}_4$.
- With a similar argument, we can show that, for any $m < n$ and $a \neq 0$, the n-sequence associated with $(x+a)^m$ cannot be extended to an $(n+1)$-sequence.

Reverse

Definition

If p is a polynomial of degree n, then we define the *reverse* of p to be the polynomial $p^*(x) = x^n p(1/x)$.

Reverse

Definition

If p is a polynomial of degree n, then we define the *reverse* of p to be the polynomial $p^*(x) = x^n p(1/x)$.

Example

Let $f(x) = 2x^5 + 7x^4 + 9x^2 - 3x + 1$. Then
$f^*(x) = x^5 - 3x^4 + 9x^3 + 7x + 2$.

Reverse

Definition

If p is a polynomial of degree n, then we define the *reverse* of p to be the polynomial $p^*(x) = x^n p(1/x)$.

Example

Let $f(x) = 2x^5 + 7x^4 + 9x^2 - 3x + 1$. Then
$f^*(x) = x^5 - 3x^4 + 9x^3 + 7x + 2$.

Proposition

Reverse preserves hyperbolicity of polynomials and the sign and multiplicity of their nonzero zeros.

Integral Representation

Proposition (Craven and Csordas)

If g_n and g_{n+1} are the Jensen polynomials associated with $\{\gamma_k\}_{k=0}^{n}$ and $\{\gamma_k\}_{k=0}^{n+1}$, respectively, then

$$g_{n+1}^{*'}(x) = (n+1)g_n^*(x)$$

Integral Representation

Proposition (Craven and Csordas)

If g_n and g_{n+1} are the Jensen polynomials associated with $\{\gamma_k\}_{k=0}^{n}$ and $\{\gamma_k\}_{k=0}^{n+1}$, respectively, then

$$g_{n+1}^{*\prime}(x) = (n+1)g_n^*(x)$$

Corollary

If g_n is a Jensen polynomial, then the associated n-sequence $\{\gamma_k\}_{k=0}^{n}$ is extendable to CMS_{n+1} if and only if there exists an $a \in \mathbb{R}$ such that

$$f(x) = (n+1)\int_a^x g_n^*(t)dt \in \mathcal{H}_{n+1}$$

In this case, $\{\gamma_k\}_{k=0}^{n} \cup \{-G(a)\}$ is an $(n+1)$-sequence, where G is an antiderivative of f with $G(0) = 0$.

New Results

Theorem (H-P)

Suppose $g_n(x) = (x+a)^m q(x)$, *where* $\deg q < n$, $a \neq 0$, *and* $m \geq 2$. *Let* $\{a_k\}$ *denote the zeros* p *and assume that* $|a| \geq |a_k|$ *or* $|a| \leq |a_k|$ *for all* $1 \leq k \leq \deg p$. *Then* $g_{n+1} \notin \mathcal{H}_{n+1}$.

New Results

Theorem (H-P)

Suppose $g_n(x) = (x+a)^m q(x)$, *where* $\deg q < n$, $a \neq 0$, *and* $m \geq 2$. *Let* $\{a_k\}$ *denote the zeros* p *and assume that* $|a| \geq |a_k|$ *or* $|a| \leq |a_k|$ *for all* $1 \leq k \leq \deg p$. *Then* $g_{n+1} \notin \mathcal{H}_{n+1}$.

Theorem (H-P; Necessary Condition for Extendability)

Suppose $g_n(x) = (x+a)^2 q(x)$, *where* $a \neq 0$, $q \in \mathcal{H}_n$, *and* $\deg g_n = m < n$. *If* $g_{n+1} \in \mathcal{H}_{n+1}$, *then* $g_{n+1}(-a) = 0$

New Results

Theorem (H-P)

Suppose $g_n(x) = (x+a)^m q(x)$, *where* $\deg q < n$, $a \neq 0$, *and* $m \geq 2$. *Let* $\{a_k\}$ *denote the zeros* p *and assume that* $|a| \geq |a_k|$ *or* $|a| \leq |a_k|$ *for all* $1 \leq k \leq \deg p$. *Then* $g_{n+1} \notin \mathcal{H}_{n+1}$.

Theorem (H-P; Necessary Condition for Extendability)

Suppose $g_n(x) = (x+a)^2 q(x)$, *where* $a \neq 0$, $q \in \mathcal{H}_n$, *and* $\deg g_n = m < n$. *If* $g_{n+1} \in \mathcal{H}_{n+1}$, *then* $g_{n+1}(-a) = 0$

Theorem (H-P; Sufficient Condition for Extendability)

Suppose $g_n(x) = (x+a)^j q(x)$, *where* $j \geq 2$, $a \neq 0$, $q \in \mathcal{H}_n$, *and* $\deg g_n = m < n$. *If* $g_{n+1}(-a) = 0$ *and* $m - j \leq 2$, *then* $g_{n+1} \in \mathcal{H}_{n+1}$.

A Necessary Condition for Extendability

Theorem

Suppose $g_n(x) = (x+a)^2 q(x)$ *where* $a \neq 0, q \in \mathcal{H}_n$, *and* $\deg g_n = m < n$. *Then if* $g_{n+1}(-a) = 0$, $g_{n+1} \in \mathcal{H}_n$.

A Necessary Condition for Extendability

Theorem

Suppose $g_n(x) = (x+a)^2 q(x)$ *where* $a \neq 0, q \in \mathcal{H}_n$, *and* $\deg g_n = m < n$. *Then if* $g_{n+1}(-a) = 0$, $g_{n+1} \in \mathcal{H}_n$.

Proof.

Let $s = 1/a$. By the integral representation and Taylor's theorem,

$$g^*_{n+1}(x) = (n+1)\int_0^x g^*_n(t)dt = \frac{n+1}{s^2}\int_0^x t^{m-n}(t+s)^2 q^*(t)dt$$

A Necessary Condition for Extendability

Theorem

Suppose $g_n(x) = (x+a)^2 q(x)$ *where* $a \neq 0, q \in \mathcal{H}_n$*, and* $\deg g_n = m < n$*.*
Then if $g_{n+1}(-a) = 0$*,* $g_{n+1} \in \mathcal{H}_n$ *.*

Proof.

Let $s = 1/a$. By the integral representation and Taylor's theorem,

$$g^*_{n+1}(x) = (n+1)\int_0^x g^*_n(t)dt = \frac{n+1}{s^2}\int_0^x t^{m-n}(t+s)^2 q^*(t)dt$$

$$g^*_{n+1}(x) = g^*_{n+1}(-s) + \sum_{k=3}^{n+1} \frac{g^{*(k)}_{n+1}(-s)}{k!}(x+s)^k$$

A Necessary Condition for Extendability

Theorem

Suppose $g_n(x) = (x+a)^2 q(x)$ *where* $a \neq 0, q \in \mathcal{H}_n$, *and* $\deg g_n = m < n$. *Then if* $g_{n+1}(-a) = 0$, $g_{n+1} \in \mathcal{H}_n$.

Proof.

Let $s = 1/a$. By the integral representation and Taylor's theorem,

$$g^*_{n+1}(x) = (n+1)\int_0^x g^*_n(t)dt = \frac{n+1}{s^2}\int_0^x t^{m-n}(t+s)^2 q^*(t)dt$$

$$g^*_{n+1}(x) = g^*_{n+1}(-s) + \sum_{k=3}^{n+1} \frac{g^{*(k)}_{n+1}(-s)}{k!}(x+s)^k$$

Thus if $g^*_{n+1}(-s) \neq 0$, then g^*_{n+1} is not hyperbolic, which implies $g_{n+1} \neq \mathcal{H}_n$. □

A Sufficient Condition for Extendability

Theorem

Suppose $g_n(x) = (x+a)^j q(x)$*, where* $j \geq 2, a \neq 0, q \in \mathcal{H}_n$*, and* $\deg g_n = m < n$*. If* $g_{n+1}(-a) = 0$ *and* $m - j \leq 2$*, then* $g_{n+1} \in \mathcal{H}_n$*.*

A Sufficient Condition for Extendability

Theorem

Suppose $g_n(x) = (x+a)^j q(x)$, *where* $j \geq 2, a \neq 0, q \in \mathcal{H}_n$, *and* $\deg g_n = m < n$. *If* $g_{n+1}(-a) = 0$ *and* $m - j \leq 2$, *then* $g_{n+1} \in \mathcal{H}_n$.

Proof.

Let $s = 1/a$. By Taylor's theorem,

$$g^*_{n+1}(x) = (x+s)^{j+1} \sum_{k=j+1}^{n+1} \frac{g^{*(k)}_{n+1}(-s)}{k!} x^{k-j-1}$$

A Sufficient Condition for Extendability

Theorem

Suppose $g_n(x) = (x+a)^j q(x)$, where $j \geq 2, a \neq 0, q \in \mathcal{H}_n$, and $\deg g_n = m < n$. If $g_{n+1}(-a) = 0$ and $m - j \leq 2$, then $g_{n+1} \in \mathcal{H}_n$.

Proof.

Let $s = 1/a$. By Taylor's theorem,

$$g^*_{n+1}(x) = (x+s)^{j+1} \sum_{k=j+1}^{n+1} \frac{g^{*(k)}_{n+1}(-s)}{k!} x^{k-j-1}$$

$$g^*_{n+1}(x) = x^{m-n+1} \sum_{k=m-n+1}^{n+1} \frac{g^{*(k)}_{n+1}(0)}{k!} x^{k-j-1}$$

A Sufficient Condition for Extendability

Theorem

Suppose $g_n(x) = (x+a)^j q(x)$*, where* $j \geq 2, a \neq 0, q \in \mathcal{H}_n$*, and* $\deg g_n = m < n$*. If* $g_{n+1}(-a) = 0$ *and* $m - j \leq 2$*, then* $g_{n+1} \in \mathcal{H}_n$*.*

Proof.

Let $s = 1/a$. By Taylor's theorem,

$$g^*_{n+1}(x) = (x+s)^{j+1} \sum_{k=j+1}^{n+1} \frac{g^{*(k)}_{n+1}(-s)}{k!} x^{k-j-1}$$

$$g^*_{n+1}(x) = x^{m-n+1} \sum_{k=m-n+1}^{n+1} \frac{g^{*(k)}_{n+1}(0)}{k!} x^{k-j-1}$$

Hence, $g^*_{n+1}(x)$ has at most $(n+1) - (j+1) - (n-m+1) = (m-j) - 1$ non-real roots. The condition $m - j \leq 2$ ensures that $g^*_{n+1}(x)$ must have all real roots.

Counterexample

In the sufficient condition, can we do any better than $m - j \leq 2$? The following result shows that we cannot:

Counterexample

In the sufficient condition, can we do any better than $m - j \leq 2$? The following result shows that we cannot:

Proposition (H-P)

There exists a function $g_n(x) = (x+a)^j q(x)$, where $j \geq 2, a \neq 0, q \in \mathcal{H}_n$, and $\deg g_n = m < n$, such that $g_{n+1}(-a) = 0$, but $g_{n+1} \notin \mathcal{H}_{n+1}$.

Counterexample

In the sufficient condition, can we do any better than $m - j \leq 2$? The following result shows that we cannot:

Proposition (H-P)

There exists a function $g_n(x) = (x+a)^j q(x)$*, where* $j \geq 2, a \neq 0, q \in \mathcal{H}_n$*, and* $\deg g_n = m < n$*, such that* $g_{n+1}(-a) = 0$*, but* $g_{n+1} \notin \mathcal{H}_{n+1}$*.*

Proof.

If $g_6(x) = \left(\frac{1}{5}x+1\right)\left(\frac{15}{21}x+1\right)\left(\frac{1}{2}x+1\right)(x+1)^2$, then $g_7(x) = \frac{1}{60}(x+1)^3(15x^2+59x+60)$. The quadratic term is irreducible over $\mathbb{R}$, hence $g_7 \notin \mathcal{H}_7$.

□

Counterexample

In the sufficient condition, can we do any better than $m - j \leq 2$? The following result shows that we cannot:

Proposition (H-P)

There exists a function $g_n(x) = (x+a)^j q(x)$*, where* $j \geq 2, a \neq 0, q \in \mathcal{H}_n$*, and* $\deg g_n = m < n$*, such that* $g_{n+1}(-a) = 0$*, but* $g_{n+1} \notin \mathcal{H}_{n+1}$*.*

Proof.

If $g_6(x) = \left(\frac{1}{5}x+1\right)\left(\frac{15}{21}x+1\right)\left(\frac{1}{2}x+1\right)(x+1)^2$, then $g_7(x) = \frac{1}{60}(x+1)^3(15x^2+59x+60)$. The quadratic term is irreducible over $\mathbb{R}$, hence $g_7 \notin \mathcal{H}_7$.

□

Note that for the example given in the proof, $j = 2$ and $m = 5$, so $m - j = 3$.

Further Research

The previous result shows that the necessary condition is not sufficient.

Further Research

The previous result shows that the necessary condition is not sufficient.

Problem

What is a necessary and sufficient condition for extendability of n-sequences which end in 0?

Further Research

The previous result shows that the necessary condition is not sufficient.

Problem

What is a necessary and sufficient condition for extendability of n-sequences which end in 0?

Problem

Under what conditions will an n-sequence of nonzero terms be non-extendable to an $(n+1)$-sequence?

Acknowledgments

We would like to thank:

- The California State University, Fresno and the National Science Foundation for their financial support (NSF Grant #DMS-1460151)
- The California State University, Fresno Mathematics REU program, and
- Our mentor, Dr. Forgács for his support.

THANK YOU!