Finite-Dimensional Hyperbolicity Preserving Operators and their Extensions

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April 2, 2024

1 Introduction

Let $\mathcal{H}$ denote the set of real univariate polynomials with only real zeroes. If $p \in \mathcal{H}$, we say that p is *hyperbolic*. Now consider a linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$. If $T[\mathcal{H}] \subset \mathcal{H}$, we say that T is *hyperbolicity preserving*. In [3], Polyá and Schur call a sequence $\{\gamma_k\}_{k=0}^{\infty} \subset \mathbb{R}$ a *classical multiplier sequence* if $\sum_{k=0}^{n} \gamma_k a_k x^k \in \mathcal{H}$ whenever $\sum_{k=0}^{n} a_k x^k \in \mathcal{H}$. Note that if a linear operator $\Gamma : \mathbb{R}[x] \to \mathbb{R}[x]$ is diagonal with respect to the standard basis $\{x^k\}_{k=0}^{\infty}$ with eigenvalues $\{\gamma_k\}_{k=0}^{\infty}$, then for any polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, $\Gamma[p(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k$. We can thus define a classical multiplier sequence equivalently as the sequence of eigenvalues of a diagonal hyperbolicity preserving operator on $\mathbb{R}$ with respect to the standard basis, i.e.

Definition 1. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers and let $\Gamma : \mathbb{R}[x] \to \mathbb{R}[x]$ be a diagonal linear operator with respect to the standard basis with eigenvalues $\{\gamma_k\}_{k=0}^{\infty}$. Then we write $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ and say that, if $\Gamma[p] \in \mathcal{H}$ for all $p \in \mathcal{H}$, then $\{\gamma_k\}_{k=0}^{\infty}$ is a classical multiplier sequence.*

Polyá and Schur gave a complete characterization of classical multiplier sequences in terms of entire functions. In order to state their result, we define the Laguerre-Polyá class, $\mathcal{L} - \mathcal{P}^{sa}$, to be the set of entire functions of the form

$$\phi(x) = cx^n e^{ax} \prod_{k=1}^{w} \left(1 + \frac{x}{x_k}\right)$$

where $c \in \mathbb{R}$, $0 \leq w \leq \infty$, $n \in \mathbb{N}$, $\sum_{k=0}^{\infty} 1/x_k < \infty$, and one of the following additional conditions hold:

- $b \geq 0$ and $x_k > 0$ for all $k \in \mathbb{N}$
- $b \leq 0$ and $x_k < 0$ for all $k \in \mathbb{N}$.

It is known [4, Theorem 44, p. 17] that $\mathcal{L} - \mathcal{P}^{sa}$ consists of precisely those entire functions which can be locally uniformly approximated by polynomials with only real zeroes of the same sign. Furthermore, the first additional condition above corresponds to those which are local uniform limits of polynomials with non-negative real zeroes, whereas the second corresponds to those which are local uniform limits of polynomials with non-positive real zeroes.

In terms of the Laguerre-Polyá class, Polyá and Schur proved the following two characterizations of classical multiplier sequences:

Theorem 2. [3] *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of non-negative real numbers. Then the following are equivalent:*

1. $\{\gamma_k\}_{k=0}^{\infty}$ *is a classical multiplier sequence.*
2. *For all* $n \in \mathbb{N}$,
$$\sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \in \mathcal{L} - \mathcal{P}^{sa} \cap \mathbb{R}_n[x]$$
3. $$\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L} - \mathcal{P}^{sa}$$

Respectively, conditions 2 and 3 are known as the *algebraic* and *transcendental characterizations* of classical multiplier sequences. We now state the following important analogues of some classical results:

Lemma 3. [2] *Let* $\{\gamma_k\}_{k=0}^{n}$ *be an n-sequence. Then the following hold:*

1. *The nonzero terms of* $\{\gamma_k\}_{k=0}^{n}$ *are either all of the same sign or of alternating signs.*
2. *If* $\gamma_i \gamma_j \neq 0$ *for any* $i < j$*, then* $\gamma_k \neq 0$ *for all* $i < k < j$.
3. $\{\gamma_{n-k}\}_{k=0}^{n}$ *is an n-sequence.*

We also note the following well-known property of hyperbolic polynomials which will be of use to us later:

Lemma 4. [3] *A polynomial of the form* $f(x) = \sum_{k=0}^{m} a_k x^k + \sum_{k=m+j}^{n} a_k x^k$ *for* $j \geq 3$ *is not hyperbolic.*

The theory of classical multiplier sequences, along with their analogues in other polynomial bases, has been developed greatly following the work of Polyá and Schur. By comparison, their analogues for finite-dimensional vector spaces $\mathbb{R}_n[x]$ have not been very thoroughly studied. In this paper, we build on the work by Craven and Csordas [2] to further develop this theory. For brevity, we introduce the notation $\mathcal{H}_n = \mathcal{L} - \mathcal{P}^{sa} \cap \mathbb{R}_n[x]$.

Definition 5. *Let* $\Gamma = \{\gamma_k\}_{k=0}^{n}$ *be a sequence of real numbers. If* $\Gamma[p] \in \mathcal{H}_n$ *for all* $p \in \mathcal{H}_n$*, then we say that* $\{\gamma_k\}_{k=0}^{n}$ *is an n-sequence.*

We now prove the natural analogue of the algebraic characterization, which will be essential to the following development.

Theorem 6. [2] *If* $\{\gamma_k\}_{k=0}^{n} \in \mathbb{R}^n$*, then* $\{\gamma_k\}_{k=0}^{n}$ *is an n-sequence if and only if*

$$\sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \in \mathcal{H}_n$$

First, we state the following theorem as a lemma:

Lemma 7 (Malo-Schur-Szegö)**.** [1] *Let*

$$A(z) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k \quad \textit{and} \quad B(z) = \sum_{k=0}^{n} \binom{n}{k} b_k x^k$$

and set

$$C(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k x^k$$

If the zeroes of $A(z)$ all lie in the interval $(-a, a)$ and the zeroes of $B(z)$ all lie in either the interval $(-b, 0)$ or the interval $(0, b)$, for $a, b > 0$, then the zeroes of $C(z)$ all lie in the interval $(-ab, ab)$.

Note that, in particular, if $A(z)$ has only real zeroes and $B(z)$ has only real zeroes of the same sign, then $C(z)$ has only real zeroes. We now proceed to the proof of the theorem:

Proof. Let $\Gamma : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ be diagonal over the standard basis with eigenvalues $\{\gamma_k\}_{k=0}^n$, i.e. $\Gamma[x^k] = \gamma_k x^k$ for all $0 \leq k \leq n$. By the binomial theorem,

$$\Gamma[(1+x)^n] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$$

Now, assume $\{\gamma_k\}_{k=0}^n \in CMS_n$. Then Γ is hyperbolicity preserving. It is clear that $\Gamma[(1+x)^n]$ has only non-positive or non-negative zeroes according to whether $\operatorname{sgn} \gamma_k = \operatorname{sgn} \gamma_{k+1}$ or $\operatorname{sgn} \gamma_k = -\operatorname{sgn} \gamma_{k+1}$. This exhausts all cases. Thus, $\Gamma[(1+x)^n] \in \mathcal{H}_n$. This gives us the desired result.

Conversely, assume $\Gamma[(1+x)^n] \in \mathcal{H}_n$. Let $p(x) \in \mathcal{H}_n$, where

$$p(x) = \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} \binom{n}{k} b_k x^k$$

Let $q(x) = \Gamma[p(x)]$. Then we have

$$q(x) = \sum_{k=0}^{n} a_k \gamma_k x^k = \sum_{k=0}^{n} \binom{n}{k} b_k \gamma_k x^k$$

From the Malo-Schur-Szegö theorem, letting $B(x) = p(x)$ and $A(x) = \Gamma[(1+x)^n]$, it follows that $q(x)$ has only real zeroes. Therefore, Γ is hyperbolicity preserving, and the result follows.

□

We note that the functions $g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$ are known as the Jensen polynomials associated with the sequence $\{\gamma_k\}_{k=0}^n$. We will use this terminology throughout the paper and denote it by $g_n(x) \sim \{\gamma_k\}_{k=0}^n$.

2 Topology of n-sequences

The algebraic characterization furnishes a very useful connection between the set of n-sequences and the set of Jensen polynomials in $\mathcal{H}_n$, which is explicated in the following theorem.

Theorem 8. *Let CMS_n denote the set of n-sequences as a topological subspace of $\mathbb{R}^{n+1}$ with the Euclidean topology, and let $\mathcal{H}_n$ denote the same set as a topological subspace of $\mathbb{R}_n[x]$ with the compact convergence topology. Let $\phi : \mathcal{H}_n \to CMS_n$ be given by*

$$\phi\left(\sum_{k=0}^{n}\binom{n}{k}\gamma_k x^k\right) = \{\gamma_k\}_{k=0}^{n}$$

Then ϕ is a homeomorphism.

Proof. Since polynomials are uniquely determined by their coefficients, it is clear that this map is invertible. Thus, to show it is a homeomorphism, it suffices to show that

$$\begin{aligned}\phi\left(\lim_{j\to\infty}\sum_{k=0}^{m}\binom{m}{k}\gamma_{k,j}x^k\right) &= \lim_{j\to\infty}\phi\left(\sum_{k=0}^{m}\binom{m}{k}\gamma_{k,j}x^k\right)\\ &= \lim_{j\to\infty}\{\gamma_{k,j}\}_{k=0}^{m}\end{aligned}$$

Let $\gamma_k = \lim_{j\to\infty}\gamma_{k,j}$ for each $0 \le k \le m$. We know that

$$\lim_{j\to\infty}\sum_{k=0}^{m}\binom{m}{k}\gamma_{k,j}x^k = \sum_{k=0}^{m}\binom{m}{k}\gamma_k x^k$$

and

$$\lim_{j\to\infty}\{\gamma_{k,j}\}_{k=0}^{m} = \{\gamma_k\}_{k=0}^{m}$$

The result follows.

□

Thus, any topological property of CMS_n can be understood as a property of $\mathcal{H}_n$, and vice versa. We now demonstrate some basic topological properties of these spaces, after stating a few lemmas which will be of use to us. The first lemma is a special case of Dini's theorem:

Lemma 9. *If $\{p_n\}$ is a sequence of polynomials which converges monotonically pointwise to a polynomial p, then convergence is also locally uniform.*

Thus, monotone pointwise convergence $p_n \to p$ is sufficient to establish $\|p_n - p\| \to 0$. In this case, we simply write $p_n \to p$.

We now proceed to provide a complete characterization of the boundary of CMS_n for each n in terms of the corresponding function space $\mathcal{H}_n$.

Theorem 10. *Let $\{\gamma_k\}_{k=0}^n$ be an n-sequence with $g_n \sim \{\gamma_k\}_{k=0}^n$, for $n \geq 2$. $\{\gamma_k\}_{k=0}^n$ is a boundary point of CMS_n if and only if either $g_n(0) = 0$ or g_n has a zero of multiplicity $m \geq 2$.*

Remark 11. We exclude the cases $n = 0$ and $n = 1$, since these spaces contain no boundary points.

Proof. First, assume g_n has a zero of multiplicity $m \geq 2$. Call this zero $-c$. Then we can write $g_n(x) = (x + c)^2 r(x)$, for some $r \in \mathcal{H}_{n-2}$. It suffices to find a sequence of polynomials in $\mathbb{R}_n[x] - \mathcal{H}_n$ which converges to g_n. Consider

$$q_m(x) = \left(x + c + \frac{i}{m}\right)\left(x + c - \frac{i}{m}\right) r(x)$$

Then, for each $m \in \mathbb{Z}_+$, q_m has two non-real zeroes which occur in conjugate pairs. Thus, it is clear that $q_m \in \mathbb{R}_n[x] - \mathcal{H}_n$ for each $m \in \mathbb{Z}_+$. Now, note that

$$\left(x + c + \frac{i}{m}\right)\left(x + c - \frac{i}{m}\right) = \left(x^2 + 2cx + c^2 - \frac{1}{m}\right)$$

Thus, q_m is monotonic in m, and it follows that $q_m \to g_n$.

Now assume $g_n(0) = 0$. Then we can write $g_n(x) = xr(x)$ for some $r \in \mathcal{H}_{n-1}$. Consider

$$q_m(x) = \left(x \pm \frac{1}{m}\right) r(x)$$

where the binomial factor has a plus or minus according to whether g_n has only non-negative or non-positive zeroes. Then it is clear that $q_m \in \mathbb{R}_n[x] - \mathcal{H}_n$ for each $m \in \mathbb{Z}_+$ and $q_m \to g_n$.

Conversely, assume that $g_n \in \mathcal{H}_n$ is a boundary point. Then there exists some sequence $\{q_m\}_{m=1}^{\infty}$ such that $q_m \in \mathbb{R}_n[x] - \mathcal{H}_n$ for all $m \in \mathbb{Z}_+$ and $q_m \to g_n$. Let

$$q_m(x) = \prod_{k=1}^{N} (x + a_{k,m})$$

Assume that $g_n(0) \neq 0$ and g_n has only simple zeroes. Then we can write

$$g_n(x) = \prod_{k=1}^{N} (x + a_k)$$

where $|a_k| > 0$ for all $1 \leq k \leq N$, $a_i = a_j$ if and only if $i = j$, and $\operatorname{sgn} a_i = \operatorname{sgn} a_j$ for all $1 \leq i \leq j \leq N$. Then, by Hurewitz's theorem, if $q_m \to g_n$, then $a_{k,m} \to a_k$ for all $1 \leq k \leq N$.

We first claim that each $a_{k,m}$ must be real. To see this, note that the non-real roots of real polynomials always occur in conjugate pairs. Thus, if some $a_{i,m}$ is non-real, it must have a conjugate pair $a_{j,m}$, i.e. $a_{j,m} = \overline{a_{i,m}}$ for some $i \neq j$. But if this is the case for all $m \in \mathbb{N}$, then we would have $\lim_{m\to\infty} a_{i,m} = \lim_{m\to\infty} a_{j,m}$, which would imply $a_i = a_j$ for some $i \neq j$, contrary to our assumption. It follows that each $a_{k,m}$ is real.

Now, similarly, we argue that each $a_{k,m}$ must have the same sign. Take $0 < \epsilon < c = \min_{1\leq k\leq N}\{|a_k|\}$ (which is possible because each is positive). Then to each $1 \leq k \leq N$, there corresponds an $N_k > 0$ such that $m > N_k$ implies $|a_{k,m} - a_k| < \epsilon < c$, which implies $a_{k,m} > 0$. Now let $M = \max_{1 \leq k \leq N}\{N_k\}$. Then $m > M$ implies $a_{k,m} > 0$ for all $1 \leq k \leq N$. It follows that $q_m \in \mathcal{H}_n$ for $m > M$, contrary to our assumption. Hence, each $a_{k,m}$ is of the same sign.

Therefore, either $g_n(0) = 0$ or g_n has a non-simple zero.

□

Taking one of many examples, e.g.

$$g_n(x) = \prod_{k=1}^{n} (x + k)$$

we have the following corollary.

Corollary 11.1. *$\mathcal{H}_n$ has nonempty interior in $\mathbb{R}_n[x]$ for all $n \in \mathbb{N}$. Equivalently, CMS_n has nonempty interior in $\mathbb{R}^{n+1}$ for all $n \in \mathbb{N}$.*

With this result, it is clear that, for $n \in \mathbb{N}_{\geq 2}$, every boundary point of $\mathcal{H}_n$ is also a boundary point of $\mathcal{H}_{n+1}$, and also that there are some boundary points of $\mathcal{H}_{n+1}$ which are not boundary points of $\mathcal{H}_n$, e.g. $(1 + x)^{n+1}$. Thus, we have $\partial\mathcal{H}_n \subset \partial\mathcal{H}_{n+1}$ properly for all $n \in \mathbb{N}_{\geq 2}$.

We now compare these with the analogous properties in the infinite case.

Theorem 12. *Let $\mathcal{L} - \mathcal{P}^{sa}$ be a topological subspace of the space of entire functions with the compact convergence topology. Then $\mathcal{L} - \mathcal{P}^{sa}$ has empty interior.*

Proof. Take any $f \in \mathcal{L} - \mathcal{P}^{sa}$. Then we have

$$f(x) = ce^{ax}x^m \prod_{k=1}^{w} \left(1 + \frac{x}{x_k}\right)$$

where $c \in \mathbb{R}$, $0 \leq w \leq \infty$, $\operatorname{sgn} a = \operatorname{sgn} x_i = \operatorname{sgn} x_j$ for all $0 \leq i \leq j \leq w$, $m \in \mathbb{N}$, and $\sum_{k=1}^{\infty} 1/x_k < \infty$. Now let

$$h_n(x) = ce^{ax - \frac{x^2}{n}}x^m \prod_{k=1}^{w} \left(1 + \frac{x}{x_k}\right)$$

Then h_n is entire but not in $\mathcal{L} - \mathcal{P}^{sa}$ for all $n \in \mathbb{Z}_+$, but $h_n \to f$ locally uniformly on $\mathbb{C}$. (To see this, note that $e^{-x^2/n} \to 1$ locally uniformly on $\mathbb{C}$.)

□

This gives us an interesting corollary.

Corollary 12.1. *Every point of $\mathcal{H}$ is a boundary point.*

Combining these results, we have

$$\emptyset = \partial\mathcal{H}_0 = \partial\mathcal{H}_1 \subset \partial\mathcal{H}_2 \subset \partial\mathcal{H}_3 \subset \cdots \subset \partial\mathcal{H} = \mathcal{H}$$

where each containment is proper.

3 Extensions of n-sequences

We now introduce the concept of an extension.

Definition 13. *Let $\{\gamma_k\}_{k=0}^n \in CMS_n$ and $\{\delta_k\}_{k=0}^m \in CMS_m$ for $m \geq n$. We say that $\{\delta_k\}_{k=0}^m$ extends $\{\gamma_k\}_{k=0}^n$, or that $\{\delta_k\}_{k=0}^m$ is an extension of $\{\gamma_k\}_{k=0}^n$, if $\{\gamma_k\}_{k=0}^n = \{\delta_k\}_{k=0}^n$.*

It is clear that, for any $\{\gamma_k\}_{k=0}^n \in CMS_n$ and any $m < n$, $\{\gamma_k\}_{k=0}^m \in CMS_m$. However, given an m-sequence, it is not clear how it can be extended to an n-sequence, for $n > m$, nor is it clear that this is always possible. Indeed, it is not, as the following example shows:

Example 14. Consider the sequence $\{1, 1.7, 2, 0\}$. The associated Jensen polynomial is

$$g_3(x) = 1 + (1.7)(3)x + (2)(3)x^2 = \frac{1}{120}\left(x + 51 + \sqrt{201}\right)\left(x + 51 - \sqrt{201}\right)$$

Thus, by the algebraic characterization, $\{1, 1.7, 2, 0\}$ is a 3-sequence. By part 2 of Lemma 3, if this sequence is extendable, then it must be extended by a 0, i.e. $\{1, 1.7, 2, 0, 0\}$ must be an extension. We now compute the associated Jensen polynomial of this new sequence:

$$g_4(x) = 1 + (1.7)(4)x + (2)(6)x^2 = \frac{1}{60}\left(x + 17 + i\sqrt{11}\right)\left(x + 17 - i\sqrt{11}\right)$$

Thus, by the algebraic characterization, $\{1, 1.7, 2, 0, 0\}$ is not a 4-sequence. It follows that $\{1, 1.7, 2, 0\}$ is not extendable.

We now seek conditions under which n-sequences are or are not extendable. In particular, we are interested in the following problem:

Problem 15 (The Extension Problem)**.** *Under what conditions can an n-sequence be extended to an m sequence for some $m > n$? If an extension is possible, then which terms will extend it?*

In [2], Craven and Csordas provide a necessary condition for extendability, namely,

Theorem 16. [2, Theorem 4.1, p. 270]. *Let $\{\gamma_k\}_{k=0}^n$ be a real sequence and let $g_n \sim \{\gamma_k\}_{k=0}^n$. If g_n has two consecutive non-simple roots, then $\{\gamma_k\}_{k=0}^n$ is not extendable to an $(n+1)$-sequence.*

This condition is, however, not sufficient. In this paper, we provide additional conditions under which n-sequences can and cannot be extended. First, we will provide some motivation for our results. The following developments abstract over the procedure in Example 14.

The aforementioned facts allow us to reformulate the problem of extending n-sequences as follows: Suppose we are given an n-sequence, $\{\gamma_k\}_{k=0}^n$. The associated Jensen polynomial g_n is given by

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$$

By the algebraic characterization, it suffices to compute g_{n+1} and check the location of its zeros. However, computing g_{n+1} requires choosing a value for γ_{n+1}. Thus, let

$$f(x,y) = yx^{n+1} + \sum_{k=0}^{n} \binom{n+1}{k} \gamma_k x^k$$

The extension problem is thus equivalent to asking if there exists a real number γ_{n+1} such that $f(x, \gamma_{n+1}) = g_n(x) \in \mathcal{H}_{n+1}$.

Now, suppose we are given an n-sequence which ends in a zero, i.e. $\{\gamma_k\}_{k=0}^n$ with $\gamma_n = 0$. Then the associated Jensen polynomial g_n has degree less than n. In particular, suppose m is the largest integer such that $\gamma_m \neq 0$. Then $m < n$, and we have

$$g_n(x) = \sum_{k=0}^{m} \binom{n}{k} \gamma_k x^k$$

By part 2 of Lemma 3, if there exists an extension of $\{\gamma_k\}_{k=0}^n$ to an $(n+1)$-sequence, then $\gamma_{n+1} = 0$. Thus, we have

$$g_{n+1}(x) = \sum_{k=0}^{m} \binom{n+1}{k} \gamma_k x^k$$

and we know that $\{\gamma_k\}_{k=0}^n$ is extendable to an $(n+1)$-sequence if and only if $g_{n+1} \in \mathcal{H}_{n+1}$.

3.1 Integral Representation

Before proceeding with our results, we provide some important background on the reverse operator.

Definition 17. *Let $p \in \mathbb{R}_n[x]$. We define $p^*(x) = x^n p(1/x)$ and call p^* the* reverse *of p with respect to n.*

The reverse operation is thus defined with respect to a natural number n, greater than or equal to the degree of the polynomial being reversed. For brevity, we will introduce the following conventions, unless otherwise specified: If p is a generic polynomial, then p^* will be used to refer to the reverse of p with respect to $\deg p$. If g_n is a Jensen polynomial, then g_n^* will be used to refer to the reverse of g_n with respect to n (regardless of the degree of g_n).

Studying the reverse polynomials rather than the original polynomials as a means of understanding n-sequences is justified by the following observations:

Lemma 18. *Let $p \in \mathbb{R}_n[x]$. Then the following hold:*

1. $(p^*)^* = p$.
2. *If $p(a) = 0$ for some $a \neq 0$, then $p^*(1/a) = 0$. Furthermore, these roots have the same multiplicity.*
3. *If $p(x) = \sum_{k=0}^n a_k x^k$, then $p^*(x) = \sum_{k=0}^n a_{n-k} x^k$.*

Thus, the reverse operator preserves the reality, sign, and multiplicity of zeros, and it can be calculated by a fairly explicit technique.

We now establish some important results about the reverse Jensen polynomials. Assume g_{n+1} is a Jensen polynomial. We compute

$$g^*_{n+1}(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} \gamma_{n+1-k} x^k$$

Differentiating, we obtain

$$\begin{aligned} g^{*'}_{n+1}(x) &= \sum_{k=0}^{n+1} \binom{n+1}{k} \gamma_{n+1-k} k x^{k-1} \\ &= (n+1) \sum_{k=0}^{n} \binom{n}{k} \gamma_{n-k} x^k \\ &= (n+1) g^*_n(x) \end{aligned}$$

We have thus proved the following:

Lemma 19. [2] *If g_n and g_{n+1} are the Jensen polynomials associated with $\{\gamma_k\}_{k=0}^n$ and $\{\gamma_k\}_{k=0}^{n+1}$, respectively, then*

$$g^{*'}_{n+1}(x) = (n+1) g^*_n(x)$$

Now, integrating both sides, we obtain

$$g^*_{n+1}(x) = (n+1) \int_a^x g^*_n(t) dt$$

for some $a \in \mathbb{R}$. We can now find a in terms of g^*_n. Let G be a primitive of g^*_n such that $G(0) = 0$. Integrating, we have $g^*_{n+1}(x) = (n+1)(G(x) - G(a))$. It follows that $g^*_{n+1}(0) = -(n+1)G(a)$. Reversing both sides shows that the leading coefficient of g_{n+1} is $-(n+1)G(a)$. By the algebraic characterization, this is precisely the term by which we are extending the n-sequence.

We are thus able to reformulate the extension problem in terms of the integral of the reverse Jensen polynomial as follows:

Proposition 20 (The Integral Representation of the Extension Problem)**.** *If g_n is a Jensen polynomial, then the associated n-sequence $\{\gamma_k\}_{k=0}^n$ is extendable to an $(n+1)$-sequence $\{\gamma_k\}_{k=0}^{n+1}$ if and only if there exists an $a \in \mathbb{R}$ such that*

$$f(x) = \int_a^x g^*_n(t) dt \in \mathcal{H}_{n+1}$$

In this case, $\gamma_{n+1} = -(n+1)G(a)$, and the associated Jensen polynomial is given by $g_{n+1}(x) = (n+1)f^*(x)$.

Now consider the case where $\deg g_n < n$. Then $\deg g_{n+1} < n$ as well. Thus, if there is an $(n+1)$-sequence $\{\gamma_k\}_{k=0}^{n+1}$ associated with g_{n+1}, we must have $\gamma_{n+1} = 0$. Since $G(0) = 0$ by construction, it suffices to take $a = 0$. Furthermore, since extension of an n-sequence by a zero is unique, taking $a = 0$ covers all possible extensions. It therefore suffices to consider the case where $a = 0$ when studying the extensions of n-sequences which end in zero.

4 New Results

We are now in a position to prove two new results on extendability. First, we state a necessary condition for extendability.

Theorem 21 (Necessary Condition for Extendability)**.** *Let* $\{\gamma_k\}_{k=0}^n$ *be a real sequence and let* $g_n \sim \{\gamma_k\}_{k=0}^n$*. Suppose* $g_n(x) = (x+a)^j q(x)$*, where* $a \neq 0$*,* $j \geq 2$*,* $q \in \mathcal{H}_n$*, and* $\deg g_n = m < n$*. If* $g_{n+1} \in \mathcal{H}_n$*, then* $g_{n+1}(-a) = 0$*.*

Proof. Without loss of generality, take $j = 2$ (higher multiplicities of the root at $-a$ can be subsumed in q). Now, with the integral representation in mind, we compute the reverse of the nth Jensen polynomial.

$$g_n^*(x) = x^{n-m}(ax+1)^2 q^*(x)$$

We are now able to apply the integral representation to get the following:

$$g_{n+1}^*(x) = (n+1)\int_0^x g_n^*(t)dt = \frac{n+1}{s^2}\int_0^x t^{n-m}(t+s)^2 q^*(t)dt$$

where $s = 1/a$. (Note that we integrate from 0 because $\deg g_n < n$.) We now expand g_{n+1}^* in a Taylor series about $x = -s$.

$$g_{n+1}^*(x) = \sum_{k=0}^{n+1} \frac{g_{n+1}^{*(k)}(-s)}{k!}(x+s)^k$$

By the integral representation, it is clear that $g_{n+1}^{*(k)}(-s) = 0$ for $k = 1, 2$. Hence, we have

$$g_{n+1}^*(x) = g_{n+1}^*(-s) + \sum_{k=3}^{n+1} \frac{g_{n+1}^{*(k)}(-s)}{k!}(x+s)^k$$

Thus, by Lemma 4, if $g_{n+1}^*(-s) \neq 0$, then g_{n+1}^* is not hyperbolic, which implies $g_{n+1} \notin \mathcal{H}_{n+1}$. Equivalent, if $g_{n+1} \in \mathcal{H}_{n+1}$, then $g_{n+1}^*(-s) = 0$, which implies $g_{n+1}(-a) = 0$.

□

We now state a sufficient condition for extendability:

Theorem 22 (Sufficient Condition for Extendability)**.** *Let* $\{\gamma_k\}_{k=0}^n$ *be a real sequence and let* $g_n \sim \{\gamma_k\}_{k=0}^n$*. Suppose* $g_n(x) = (x+a)^j q(x)$*, where* $a \neq 0$*,* $j \geq 2$*,* $q \in \mathcal{H}_n$*, and* $\deg g_n = m < n$*. If* $g_{n+1}(-a) = 0$ *and* $m - j \leq 2$*, then* $g_{n+1} \in \mathcal{H}_n$*.*

Proof. Similar to the above proof, let $s = 1/a$. We have

$$g_n^*(x) = x^{n-m}(ax+1)^j q^*(x)$$

By the integral representation, we have

$$g_{n+1}^*(x) = (n+1)\int_0^x g_n^*(t)dt = \frac{n+1}{s^j}\int_0^x t^{n-m}(t+s)^j q^*(t)dt$$

From this, it is clear that $g_{n+1}^{*(k)}(-s) = 0$ for $k = 1, 2, \ldots, j$, and that $g_{n+1}^{*(k)}(0) = 0$ for $k = 1, 2, \ldots, m - n + 1$. We now expand g_{n+1}^{*} as a Taylor series about 0 and $-s$:

$$g_{n+1}^{*}(x) = g_{n+1}^{*}(-s) + (x + s)^{j+1} \sum_{k=j+1}^{n+1} \frac{g_{n+1}^{*(k)}(-s)}{k!} x^{k-j-1}$$

and

$$g_{n+1}^{*}(x) = g_{n+1}^{*}(0) + x^{m-n+1} \sum_{k=m-n+1}^{n+1} \frac{g_{n+1}^{*(k)}(0)}{k!} x^{k-j-1}$$

We know that $g_{n+1}^{*}(0) = 0$ directly from the integral representation; hence, g_{n+1}^{*} has a root of multiplicity at least $m-n+1$ at 0. Furthermore, if we assume that $g_{n+1}^{*}(-s) = 0$, then the above shows that g_{n+1}^{*} has a root of multiplicity at least $j + 1$ at $-s$. We know from the fundamental theorem of algebra that g_{n+1}^{*} has exactly $n + 1$ roots, counting multiplicities. Hence, g_{n+1}^{*} has at most $(n + 1) - (j + 1) - (n - m - 1) = m - j - 1$ non-real roots. In particular, if $m - j \leq 2$, then g_{n+1}^{*} has at most 1 non-real root. However,the non-real roots of real polynomials must come in conjugate pairs; hence, under this assumption, g_{n+1}^{*} is hyperbolic.

Finally, the integral representation shows that g_{n+1}^{*} does not have any roots on the positive real axis. It follows that $g_{n+1}^{*} \in \mathcal{H}_n$, which implies $g_{n+1} \in \mathcal{H}_n$. $\square$

Together, these theorems provide new necessary and sufficient conditions for the extendability of n-sequences.

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