Group Actions on Products of Spheres

Kendra Plante

1 Introduction

In this paper, we explore free group actions using the techniques of group cohomology. In particular, we are interested in free actions of finite abelian groups on products of spheres. We start by providing some basic examples which motivate our general course of study.

Proposition 1. *Let $G = (\mathbb{Z}/p)^r$. If G acts freely on S^1, then $r = 1$.*

Proof. Assume G acts freely on S^1. Then we have a covering $p : S^1 \to S^1/G$. It follows from covering space theory that $G \cong \pi_1(S^1/G)/p_*(\pi_1(S^1))$. But since G is a finite group, we know that $S^1/G \cong S^1$. Thus, we have

$$G \cong \pi_1(S^1)/p_*(\pi_1(S^1)) \cong \mathbb{Z}/|G|$$

The result follows.

□

More generally, if G acts freely on S^n, then it is known that we must have $r = 1$. This fact has led to the following conjecture:

Conjecture 1. *Let $G = (\mathbb{Z}/p)^r$. If G acts freely on $\prod_{i=1}^{k} S^{n_i}$, then $r \leq k$.*

This conjecture remains unproven, but many partial results are known. For example, in this paper, we will prove it in the case of an elementary p-group acting freely and homologically trivially on a product of equidimensional spheres, i.e. $n_i = n_j$ for all i, j. We will then explore some additional work on the conjecture in other cases. The primary tool for studying these actions will be group cohomology, specifically Tate cohomology.

2 Algebraic Foundations

Let G be a group. We wish to construct the notion of a G-module. In order to do this, we impose a natural ring structure on G: Let $\mathbb{Z}G$ be the free $\mathbb{Z}$-module generated by the elements of G, i.e. the abelian group consisting of elements $\sum_{g \in G} a(g)g$ where $a(g) \in \mathbb{Z}$ and $a(g) = 0$ for almost all g, with multiplication induced by multiplication in G. We call $\mathbb{Z}G$ the *group ring* of G.

The naturality of the construction $G \mapsto \mathbb{Z}G$ is characterized by the following adjunction:

Theorem 1. *Let $U : \mathbf{Rng} \to \mathbf{Grp}$ be the functor which maps each ring (with unity) to its group of units. Then the functor $\mathbb{Z}(-) : \mathbf{Grp} \to \mathbf{Rng}$, $G \mapsto \mathbb{Z}G$, which maps each group to its group ring, is a left adjoint of U.*

Proof. Let G be a group and R a ring with unity. Let $f : G \to U(R)$ be a group homomorphism. Define $f' : \mathbb{Z}G \to R$ by extending f linearly, i.e.

$$f'\left(\sum_i c_i x\right) = \sum_i c_i f(x)$$

for $c_i \in \mathbb{Z}$ (only finitely many nonzero) and $x \in G$. Multiplication in $\mathbb{Z}G$ is given by the operation in G, so f' preserves multiplication by assumption. Thus, f' is a ring homomorphism. Thus, we can extend each group homomorphism $f : G \to U(R)$ to a ring homomorphism $f' : \mathbb{Z}G \to R$. It is clear that we can also restrict ring homomorphisms $f : \mathbb{Z}(G) \to R$ to group homomorphisms $f'' : G \to U(R)$, and that these processes are inverses, i.e. $f' \circ f'' = \mathrm{id}_{\mathbb{Z}G}$ and $f'' \circ f' = id_G$. Thus, we have a bijection

$$\mathrm{Hom}(\mathbb{Z}G, R) \cong \mathrm{Hom}(G, U(R))$$

Naturality follows from the fact that $\mathbb{Z}(f(G)) = f'(\mathbb{Z}(G))$ and $U(f(R)) = f''(U(R))$. This establishes the desired adjunction.

□

We refer to modules over $\mathbb{Z}G$, for any group G, as G-modules. In particular, a (left) G-module is an abelian group M with a (left) G-action on M, i.e. a ring homomorphism $\mathbb{Z}G \to \mathrm{End}\, M$. For example, we may consider $\mathbb{Z}$ a trivial Gmodule, namely with the trivial action of $\mathbb{Z}G$ on Z.

Throughout this paper, we will use G as a subscript to refer to the group ring $\mathbb{Z}G$, e.g. $\mathrm{Hom}_G(A, B)$ will refer to the group of $\mathbb{Z}G$-module homomorphisms from A to B. We will also refer to a complex of G-modules as a *G-complex.* With these foundations in place, we are ready to define the cohomology of a group.

Definition 1. *Let G be a group and A a G-module. Let*

$$H^n(G, M) = \mathrm{Ext}^n_G(\mathbb{Z}, M) = R^n(\mathrm{Hom}_G(-, M))$$

$$H_n(G, M) = \mathrm{Tor}^n_G(\mathbb{Z}, M) = L^n(- \otimes_G M)$$

where $\mathbb{Z}$ denotes the trivial G-module. We call $H^n(G, M)$ the nth cohomology of G with coefficients in M, and $H_n(G, M)$ the nth homology of G with coefficients in M.

Explictly, the construction of the cohomology groups proceeds as follows: Form a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$

$$P : \ \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

i.e. an exact sequence in which P_k is projective as a G-module for all k. Now, form the cochain complex

$$\mathrm{Hom}_G(P, M): \ 0 \to \mathrm{Hom}_G(P_0, M) \to \mathrm{Hom}_G(P_1, M) \to \dots$$

where we dualize P and remove the $\mathbb{Z}$ term. The nth homology of this complex is the $\mathrm{Ext}^n_G(\mathbb{Z}, M)$, i.e. the nth cohomology of G with coefficients in M. Note that any two projective resolutions of a module are chain homotopic, and chain homotopies induce isomorphisms in homology and cohomology, hence this construction is well-defined.

Similarly, we can compute the homology groups by forming the chain complex

$$P \otimes_G M : \dots \to P_1 \otimes_G M \to P_0 \otimes_G M \to 0$$

and taking its nth homology.

However, this construction is still too abstract to use to compute the homology and cohomology groups in specific cases. We provide an example of one computational method called the *bar resolution*, which gives a concrete procedure for constructing projective resolutions over $\mathbb{Z}G$. For each $n \in \mathbb{N}$, let F_n be the free G-module generated by the $(n+1)$-tuples $(g_0, g_1, \dots, g_n)$ of elements of G. Define a left G-action on F_n by

$$g \mapsto ((g_0, g_1, \dots, g_n) \to (gg_0, gg_1, \dots, gg_n))$$

and a boundary operator $\partial_n : F_n \to F_{n-1}$ given by

$$\partial_n(g_0, g_1, \dots, g_n) = \sum_{k=0}^{n} (-1)^k (g_0, \dots, \hat{g_k}, \dots, g_n)$$

Finally, define the augmentation $\epsilon : F_0 \to \mathbb{Z}$ by $\epsilon = 1$. With these definitions,

$$\dots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} \mathbb{Z}$$

forms a projective resolution (in fact, a free resolution) of $\mathbb{Z}$ over $\mathbb{Z}G$. This gives us a useful algebraic procedure for computing the cohomology of a group directly from the algebraic definition given above.

3 Classifying Spaces

We now give a topological interpretation of group cohomology, which provides an alternative method for computing the cohomology of a group in terms of cell complexes. First, we introduce the notion of a topological group and fiber bundle.

Definition 2. *A topological group is a set G which is both a group and a topological space, such that the group multiplication $(a, b) \mapsto ab$ and inversion $a \mapsto a^{-1}$ are both continuous functions.*

Definition 3. *Let E and B be topological spaces, B connected, and $p : E \to B$ be a continuous surjection. Choose a point $b \in B$ and let $F = p^{-1}(b)$. If for any point $b' \in B$ there is a neighborhood $U_{b'} \subset B$ of b' and a homeomorphism $\phi : U_{b'} \times F \to p^{-1}(U_{b'})$, such that the following diagram commutes*

$$\begin{array}{ccc} U_{b'} \times F & \xrightarrow{\phi} & p^{-1}(U_{b'}) \\ & \searrow^{\pi} & \downarrow p \\ & & U_{b'} \end{array}$$

commutes, where $\pi : U_{b'} \times F \to U_{b'}$ is the natural projection, then

$$F \to E \xrightarrow{p} B$$

is called a fiber bundle, with base space B, total space, E, and fiber F.

Note that, under these conditions, $p^{-1}(x) \cong F$ for all $x \in B$, so F is independent of the choice of base point, up to homeomorphisms. We now introduce the notion of a principle bundle, which combines the notions of a topological group and a fiber bundle.

Definition 4. *Let G be a topological group and $G \to E \xrightarrow{p} B$ be a fiber bundle. Assume the following conditions hold:*

1. *G acts freely on the total space E via $\psi : E \times G \to E$ in such a way that the following diagram commutes*

$$\begin{array}{ccc} E \times G & \xrightarrow{\psi} & E \\ \downarrow{p \times 1} & & \downarrow p \\ B \times 1 & \xrightarrow{\text{id}} & B \end{array}$$

2. *The action on the fibers $p^{-1}(b) \times G \to p^{-1}(b)$ induced by psi is free and transitive.*

3. *We have the same commutative diagram as above, namely*

$$\begin{array}{ccc} U_b \times G & \xrightarrow{\phi} & p^{-1}(U_b) \\ & \searrow^{\pi} & \downarrow p \\ & & U_b \end{array}$$

   with the added condition that $\phi(xg) = \psi(\phi(x), g) = \phi(x)g$, for all $g \in G$ and $x \in U_b \times G$.

Then we call $G \to E \xrightarrow{p} B$ a principle G-bundle.

Note that the added condition on the homeomorpshism $\phi : U_b \times G \to p^{-1}(U_b)$ effectively asserts that ϕ commutes with group actions. When this condition holds of a continuous map, we call it an *equivarient map*.

A particularly interesting and important case of principle bundles in eqivarient topology is the classifying space BG of a group G. These spaces are defined in terms of pullback squares of fiber bundles. If $F \to E \xrightarrow{p} B$ is a fiber bundle and $f : X \to B$ is any continuous function, we can form the pullback square

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

Then the map $f^*(E) \to X$ induces a fiber bundle over X in terms of f and $E \xrightarrow{p} B$. According to the following theorem due to Milnor, principles G-bundles can be classified up to pullback squares as follows:

Theorem 2. *Let G be a group. Then there is a space BG and a principle G-bundle $G \to EG \to BG$, such that EG is weakly contractible, and if $G \to E \to X$ is any other principle G-bundle, then there is a unique map $f : X \to BG$, up to homotopy, such that $f^*(EG) = E$.*

In this case, BG is called the *classifying space* of G and EG the *universal G-bundle*. Now, note that for the fiber bundle $G \to EG \to BG$, we have a long exact sequence of homotopy groups

$$\cdots \to \pi_n(G) \to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \cdots \to \pi_0(EG) \to 0$$

Since EG is weakly contractible, we have $\pi_n(EG) = 0$ for all n, so the long exact sequence reduces to

$$\cdots \to 0 \to \pi_n(BG) \to \pi_{n-1}(G) \to 0 \to \cdots \to 0 \to \pi_1(BG) \to \pi_0(G) \to 0$$

It follows that $\pi_{n+1}(BG) \cong \pi_n(G)$ for all n. We know that $\pi_0(G) = G$. If G is discrete, then we can also conclude that $\pi_n(G) = 0$ for $n > 0$. In this case, we get $\pi_n(BG) = G$ for $n = 1$ and 0 otherwise. Thus, if G is discrete, then BG is exactly the Eilenberg-Maclane space $K(G, 1)$.

From the above, we can conclude the following for a general classifying space and universal G-bundle:

1. EG is also a universal cover of BG
2. $\pi_1(BG) = G$

It follows now from the theory of covering spaces that G acts on EG via Deck transformations, which in turn induces an action of G on the singular chains $C_*(EG)$. This action makes each $C_*(EG)$ into a G-module. We thus get a chain complex

$$\cdots \to C_2(EG) \to C_1(EG) \to C_0(EG) \to \mathbb{Z}$$

But since EG is weakly contractible, this complex is exact. Furthermore, since the action by G is free, this sequence forms a free (hence projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. It follows that we can use these singular chains to compute group cohomology as

$$H^n(G, M) = H^n(\mathrm{Hom}_G(C_*(EG), M))$$

However, we also know from singular cohomology that the right hand side is exactly the cohomology of BG with coefficients in M. Hence, we arrive at the following isomorphism

$$H^*(G, M) \cong H^*(BG, M)$$

Similarly, we can show that

$$H_*(G, M) \cong H_*(BG, M)$$

These isomorphisms reveal a natural connection between the algebraic description of group cohomology and the topology of classifying spaces.

4 Motivating Examples

We now provide some simple examples to motivate the group cohomological approach to the conjecture stated in the introduction. Recalling the first theorem, we know that the only finite abelian groups which act freely on S^1 are cyclic. We will now see that this action has implications for the structure of the cohomology groups.

First, note that any cellular decomposition of S^1 has cells only in dimension 0 and 1. Hence, we get a cellular chain complex

$$0 \to C_1 \to C_0 \to 0$$

To make this exact, we stick on the homology groups in dimension 0 and 1, namely

$$0 \to H_1(S^1) \to C_1 \to C_0 \to H_0(S^1) \to 0$$

We know that $H_1(S^1) = H_0(S^1) = \mathbb{Z}$, hence this forms a finite length free resolution of $\mathbb{Z}$. Now, we show how we can use a specific free group action to construct such resolutions explicitly over $\mathbb{Z}G$.

let $G = \mathbb{Z}/n$. Let t denote the generator of G. The group ring $\mathbb{Z}G$ of G is then easily seen to be $\mathbb{Z}[t]/(t^n - 1)$. Now describe an action of G on S^1 as follows: Subdivide S^1 into n vertices with n edges. Then there is an action of

G on S^1 which freely permutes these vertices and edges. Namely, the vertices and edges can then be represented by $\{t^k v\}_{k=0}^{n-1}$ and $\{t^k e\}_{k=0}^{n-1}$, respectively.

One cycle around the circle can then be represented by the sum of the edges $\sum_{k=0}^{n-1} t^k e$. This element therefore generates the first homology group, $H_1(S^1) \cong \mathbb{Z}$. Since $t^n = 1$, it is clear that G acts trivially on $H_1(S^1)$. We therefore have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\epsilon(g) = 1$ for all $g \in G$ and $\eta(1) = N = \sum_{k=0}^{n-1} t^k$, which is exactly the short exact sequence of cellular chains written above for the cellular decomposition of S^1 induced by the action of G. N is called the *norm element*. Since $\epsilon\eta(x) = Nx$ for any $x \in \mathbb{Z}G$, we can splice these short exact sequences together to get a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

$$\dots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Applying the functor $\mathrm{Hom}(-, \mathbb{Z})$, we obtain the cochain complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \dots$$

hence we obtain

$$H^i(G, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/n, & i \text{ even} \\ 0, & i \text{ odd} \end{cases}$$

Similarly, applying the functor $- \otimes \mathbb{Z}$, we obtain the chain complex

$$\dots \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

from which we compute

$$H_i(G, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/n, & i \text{ odd} \\ 0, & i \text{ even} \end{cases}$$

Finally, note that if n is prime and we compute the homology or cohomology with coefficients in $\mathbb{Z}/n$, then all boundary maps in the chain and cochain complexes are isomorphisms and we get

$$H(G, \mathbb{Z}/n) \cong \mathbb{Z}/n$$

Thus, using the canonical free action of G on S^1, we get the result that both the homology and cohomology of G (with $\mathbb{Z}$ coefficients) are 2-periodic. This construction generalizes, and we actually have the following result from *Cohomology of Groups* by Kenneth Brown (proposition 10.2), which relies on spectral sequence calculations and an application of the Lefschetz fixed point theorem.

Theorem 3. *Let G be a finite group and X a finite-dimensional free G-complex such that $H_*(X) \cong H_*(S^{2n-1})$ for some $n \in \mathbb{N}$. Then G has periodic cohomology of period $2k$.*

We can, however, see how this construction would proceed for the case $X = S^{2n-1}$ if we know that G acts trivially on $H_{2n-1}(X)$ (which requires an application on the Lefschets fixed point theorem). If we take this for granted, the construction proceeds very much as it did in the case of S^1.

Say G acts freely on S^{2n-1} and trivially on $H_{2n-1}(X)$. From basic topology, we can construct a complex of cellular chains

$$0 \to C_{2n-1} \to \cdots \to C_1 \to C_0 \to 0$$

Each C_k is free by definition. This is exact whenever homology is trivial, which in the case of S^{2n-1}, is only the case in dimensions 0 and $2n-1$. We therefore can make this exact simply by adding the 0th and $(2n-1)$st homology onto the ends

$$0 \to H_{2n-1}(X) \to C_{2n-1} \to \cdots \to C_1 \to C_0 \to H_0(X) \to 0$$

We know that $H_{2n-1}(X) = H_0(X) = \mathbb{Z}$ and we also know that the cellular chains in dimension k are free on the k cells, which in the case of the group action correspond to the elements of G. Thus, we can rewrite this as

$$0 \to \mathbb{Z} \to \mathbb{Z}G \to \cdots \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

Since the action of G on $H_{2n-1}(X)$ and $H_0(X)$ are both trivial, the $\mathbb{Z}$ on the left and right of the sequence are both trivial G-modules, hence the maps $\mathbb{Z}G \to \mathbb{Z}$ and $\mathbb{Z} \to \mathbb{Z}G$ in the above sequence can be composed to combine the sequence iteratively into a long exact sequence

$$\ldots \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

Since $\mathbb{Z}G$ is free, this forms a free (hence projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Furthermore, this resolution is periodic with period $2k$, in the sense that the sequence of boundary maps repeats every $2k$ times. This periodicitiy is clearly preserved under the functors $\mathrm{Hom}(-, M)$ and $- \otimes M$. It therefore follows that any finite group which acts freely on any odd dimensional sphere must have periodic cohomology.

This covers the case of odd-dimensional spheres. The case of even-dimensional spheres is much easier: Any map $S^{2n} \to S^{2n}$ without fixed points is homotopic to the antipodal map $x \mapsto -x$, hence is of degree -1. Since each $g \in G$ induces such a map under a free action, and degree is multiplicative, it follows that G can contain at most one non-trivial element. Hence, the only non-trivial group which acts on even-dimensional spheres is $\mathbb{Z}/2$ (which certainly has periodic cohomology, as computed above).

We thus can state in general that any finite group which acts freely on a sphere has periodic cohomology. This gives us a powerful tool for testing which groups can act freely on spheres. For example, consider $G = \mathbb{Z}/p \times \mathbb{Z}/p$ for p prime. We can use the Kunneth formula to compute the cohomology of G with $\mathbb{Z}/p$ coefficients:

$$H^k(\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z}/p) \cong \bigoplus_{i+j=k} H^i(\mathbb{Z}/p, \mathbb{Z}/p) \otimes H^j(\mathbb{Z}/p, \mathbb{Z}/p) \cong (\mathbb{Z}/p)^{k+1}$$

Thus, the cohomology is not periodic, so it follows from the above that $\mathbb{Z}/p \times \mathbb{Z}/p$ cannot act freely on any sphere. We can use arguments of this sort to show that any finite abelian group which is not cyclic cannot act freely on any sphere, as stated in the introduction.

Throughout the rest of the paper, we will introduce new notions which allow us to expand these techniques to answer similar questions about products of spheres.

5 Tate Cohomology

The homology and cohomology groups constructed above give rise to sequences $\{H_i(G, M)\}_{i \in \mathbb{N}}$ and $\{H^i(G, M)\}_{i \in \mathbb{N}}$. We introduce a similar construction which combines these two into one sequence, indexed over $\mathbb{Z}$. This is known as the *Tate cohomology*. It is especially useful in studying the cohomology of finite groups. With that said, we will henceforth assume G is a finite group.

In order to define the Tate cohomology, we extend the notion of a projective resolution to allow for sequences which extend out in both directions. These are called *complete resolutions*.

Definition 5. *A complete resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ is a long exact sequence*

$$\cdots \to F_1 \to F_0 \to F_{-1} \to \ldots$$

where each F_i is a projective G-module, together with a map $\epsilon : F_0 \to \mathbb{Z}$, called the augmentation, such that

$$\cdots \to F_2 \to F_1 \to F_0 \xrightarrow{\epsilon} \mathbb{Z}$$

is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ in the usual sense.

This definition effectively says that a complete resolution is a long exact sequence of projective G-modules which becomes a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ in sufficiently high degree. Similar to the case of projective resolutions, it can be shown that complete resolutions are unique up to chain homotopies which preserve augmentation.

We now introduce Tate cohomology in terms of complete resolutions.

Definition 6. *Let G be a finite group and let F be a complete resolution of G ove $\mathbb{Z}G$. Let*

$$\hat{H}^i(G, M) = H^i(\mathrm{Hom}_G(F, M))$$

for $i \in \mathbb{Z}$. We call $\hat{H}^i(G, M)$ the Tate cohomology of G with coefficients in M.

It is clear that $\hat{H}^i(G, M) = H^i(G, M)$ for $i > 0$. For the negative terms, suppose we have a projective resolution F of $\mathbb{Z}$ over $\mathbb{Z}G$. We can form the dual resolution $F' = \mathrm{Hom}_G(\sum F, \mathbb{Z})$ where $\sum F$ is the suspension of F, i.e. $(\sum F)_i = F_{i-1}$. Letting η denote the dual of the augmentation ϵ, this forms a long exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\eta} F'_0 \to F'_1 \to \dots$$

where each F'_i is projective and finitely generated. (Long exact sequences of this form are called *backwards projective resolutions.*) Thus, if we let $F_{-i} = F'_{i-1}$, then

$$\begin{array}{ccccccccc} \dots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & F_{-1} & \longrightarrow & \dots \\ & & & & \downarrow{\scriptstyle \epsilon} & \overset{\eta}{\nearrow} & & & \\ & & & & \mathbb{Z} & & & & \end{array}$$

forms a complete resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Moreover, we have

$$\mathrm{Hom}_G\left(\mathrm{Hom}_G\left(\sum F, \mathbb{Z}\right), M\right) \cong \sum F \otimes_G M$$

Thus, we have that $\hat{H}^i(G, M) = H_{i-1}(G, M)$ for $i < -1$. It is in this sense that Tate cohomology combines homology and cohomology groups into one sequence. We have left to describe the $\hat{H}^0$ and $\hat{H}^{-1}$ terms.

Recall that we have

$$H_0(G, M) \cong M_G = M/(m - gm), \quad m \in M, g \in G$$

$$H^0(G, M) \cong M^G = \{m \in nM \mid gm = m, \quad g \in G\}$$

Now, let $N = \sum_{g \in G} g$. We call N the *norm element* of G. Define a map $\alpha : M \to M$ by $m \mapsto Nm$ (recall that M is a G-module). Then α factors through the projection and injection maps $M \twoheadrightarrow M_G$ and $M^G \hookrightarrow M$ as follows

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & M \\ \downarrow & & \uparrow \\ M_G & \xrightarrow{\alpha'} & M^G \end{array}$$

This map $\alpha' : M_G \to M^G$ gives us a short exact exact sequence

$$0 \to \hat{H}^{-1}(G,M) \to H_0(G,M) \xrightarrow{\alpha'} H^0(G,M) \to \hat{H}^0(G,M) \to 0$$

from which we conclude that

$$\hat{H}^{-1}(G,M) = \ker \alpha'$$

$$\hat{H}^0(G,M) = \operatorname{coker} \alpha'$$

This allows us to prove the following fact which we will use in the subsequent developments:

Proposition 2. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of G-modules. Then there is a long exact sequence*

$$\cdots \to \hat{H}^i(G,A) \to \hat{H}^i(G,B) \to \hat{H}^i(G,C) \to \hat{H}^{i+1}(G,A) \to \dots$$

Proof. Note that for $i < -1$, this is exactly the usual long exact sequence for homology. Similarly, for $i > 0$, this is exactly the usual long exact sequence for cohomology. Therefore, it is enough to construct the map $\delta : \hat{H}^{-1}(G,C) \to \hat{H}^0(G,A)$, since the other maps are induced by the maps in the short exact sequence. To do this, we apply the snake lemma to the following diagram

$$\begin{array}{ccccccccc}
\dots \longrightarrow H_1(G,C) & \xrightarrow{\delta} & H_0(G,A) & \longrightarrow & H_0(G,B) & \longrightarrow & H_0(G,C) & \longrightarrow & 0 \\
 & \swarrow_{\alpha'} & & \swarrow_{\alpha'} & & \swarrow_{\alpha'} & & & \\
0 \longrightarrow H^0(G,A) & \longrightarrow & H^0(G,B) & \longrightarrow & H^0(G,C) & \xrightarrow{\delta} & H^1(G,A) & \longrightarrow & \dots
\end{array}$$

The snake lemma furnishes a map $\ker \alpha' \xrightarrow{\delta} \operatorname{coker} \alpha'$, as desired.

□

Another important property of Tate cohomology which we will make use of is a phenomenon known as *dimension shifting*. The case of this property which we will use is characterized by the following proposition

Proposition 3. *Let I_G denote the kernel of the augmentation map $\epsilon : \mathbb{Z}G \to \mathbb{Z}$. Then for each G-module A and each $i \in \mathbb{Z}$, we have*

$$H^{i+1}(G,A) \cong \hat{H}^i(G,A^*) \qquad \text{and} \qquad \hat{H}^{i-1}(G,A) \cong \hat{H}^i(G,A_*)$$

where $A^* = \operatorname{Hom}_{\mathbb{Z}}(I_G, A)$ and $A_* = I_G \otimes_{\mathbb{Z}} A$.

6 Exponents

We will now use Tate cohomology to obtain results about the exponent of homology and cohomology groups, which we will then use to obtain the initial conjecture in certain special cases. Let M be a $\mathbb{Z}$-module with torsion. We let $\exp M$ denote the least positive $n \in \mathbb{Z}$ such that $nx = 0$ for all $x \in M$, and call $\exp M$ the *exponent* of M. We begin this section with a lemma:

Lemma 1. *Assume we have a long exact sequence*

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \dots$$

such that A, B, and C have finite exponent. Then $\exp B$ divides $\exp A \exp C$.

Proof. Let $a = \exp A$, $b = \exp B$, and $c = \exp C$. Take any $y \in B$. Then by definition, $g(cy) = cg(y) = 0$, hence $cy \in \ker g$, which implies that $cy \in \operatorname{im} f$. Thus, there is some $x \in A$ such that $f(x) = cy$. But note that $af(x) = f(ax) = f(0) = 0$. Hence, $acy = 0$. Since y was arbitrary, this shows that $acy = 0$ for all $y \in B$. It follows that b divides ac, as desired. □

We are now able to prove the following important theorem due to William Browder:

Theorem 4 (Browder)**.** *Let C be a free connected G-complex of finite length. Then*

1. $|G|$ *divides* $\prod_{i=1}^{n} \exp H^{i+1}(G, H_i(C))$
2. $|G|$ *divides* $\prod_{i=1}^{n} \exp H_{i+1}(G, H^i(C))$

Proof. Consider a short exact sequence of G-modules

$$0 \to A_0 \to B \to A_1 \to 0$$

As shown above, this gives us a long exact sequence of Tate cohomology groups given by

$$\cdots \to \hat{H}^i(G, A_0) \to \hat{H}^i(G, B) \to \hat{H}^i(G, A_1) \to \hat{H}^{i+1}(G, A_0) \to \dots$$

From this, we easily see that if $\hat{H}^*(G, B) = 0$, then $\hat{H}^i(G, A_1) \cong \hat{H}^{i+1}(G, A_0)$, for then the above reduces to a family of short exact sequences of the form

$$0 \to \hat{H}^i(G, A_1) \to \hat{H}^{i+1}(G, A_0) \to 0$$

Now, form a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

$$C_n \to \cdots \to C_1 \to C_0 \xrightarrow{\epsilon} \mathbb{Z}$$

For each k, let Z_k and ∂C_k denote the kernel and image of the boundary map $C_k \to C_{k-1}$, respectively. Then for each k, we get a short exact sequence

$$0 \to Z_k \to C_k \to \partial C_k \to 0$$

immediately from the definitions. Now, observe that $\hat{H}^*(G, C_k) = 0$ for all k, since each C_k is a free G-module. Thus, it follows from the above that $\hat{H}^i(G, \partial C_k) \cong \hat{H}^{i+1}(G, Z_k)$.

Using the definition of homology, we also get a short exact sequence

$$0 \to \partial C_{k+1} \to Z_k \to H_k(C) \to 0$$

for each k. Form the long exact sequence of Tate cohomology groups

$$\cdots \to \hat{H}^i(G, \partial C_{k+1}) \to \hat{H}^i(G, Z_k) \to \hat{H}^i(G, H_k(C)) \to \hat{H}^{i+1}(G, \partial C_{k+1}) \to \ldots$$

Using the isomorphism $\hat{H}^i(G, \partial C_k) \cong \hat{H}^{i+1}(G, Z_k)$ from above, we can elimitate all the terms with coefficients in Z_k to get

$$\cdots \to \hat{H}^i(G, \partial C_{k+1}) \to \hat{H}^{i-1}(G, \partial C_k) \to \hat{H}^i(G, H_k(C)) \to \hat{H}^{i+1}(G, \partial C_{k+1}) \to \ldots$$

Using the lemma above, we conclude, for each $i > 0$ and $k \geq 0$, that $\exp \hat{H}^i(G, \partial C_k)$ divides $\exp \hat{H}^{i+1}(G, \partial C_k) \exp \hat{H}^{i+1}(G, H_k(C))$. Now, set $i = k$ and take the product of both sides over $k = 1, \ldots, n$. From this, we conclude that

$$\prod_{k=1}^{n} \frac{\exp \hat{H}^k(G, \partial C_k)}{\exp \hat{H}^{k+1}(G, \partial C_k)} \qquad \text{divides} \qquad \prod_{k=1}^{n} \exp \hat{H}^{k+1}(G, H_k(C))$$

The product on the left hand side telescopes and we get

$$\prod_{k=1}^{n} \frac{\exp \hat{H}^k(G, \partial C_k)}{\exp \hat{H}^{k+1}(G, \partial C_k)} = \frac{\exp \hat{H}^1(G, \partial C_1)}{\exp \hat{H}^{n+1}(G, \partial C_{n+1})} = \exp \hat{H}^1(G, \partial C_1)$$

The last equality holds because $C_{n+1} = 0$, which implies $\hat{H}^{n+1}(G, \partial C_{n+1} = 0$, hence $\exp \hat{H}^{n+1}(G, \partial C_{n+1}) = 1$. Thus, we conclude that

$$\exp \hat{H}^1(G, \partial C_1) \qquad \text{divides} \qquad \prod_{k=1}^{n} \exp \hat{H}^{k+1}(G, H_k(C))$$

Now, recall that the modules $\{C_k\}$ were constructed as a projective resolution of $\mathbb{Z}$, hence $\partial C_1 = \ker \epsilon : C_0 \to \mathbb{Z}$, by construction. Thus, $0 \to \partial C_1 \to C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$ is exact. Equivalently, $\partial C_1 \cong I_G$. It follows from the dimension shifting property of Tate cohomology that

$$\hat{H}^1(G, \partial C_1) \cong \hat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/|G|$$

The first part of the theorem follows.

To prove the second part, we repeat the same argument using the dual resolution

$$0 \to \mathbb{Z} \xrightarrow{\epsilon} C^0 \to C^1 \to \cdots \to C^n$$

with $C^n = \mathrm{Hom}_G(C_n, \mathbb{Z})$. Each C^n is free, since each C_n is free. Now, like above, letting Z^k and δC^k denote the kernel and image of the coboundary map $C_k \to C_{k+1}$, respectively, we get short exact sequences

$$0 \to Z^k \to C^k \to \delta C^k \to 0$$

$$0 \to \delta C^{k-1} \to Z^k \to H^k(C)$$

which yields, for each $k > 0$, the long exact sequence

$$\cdots \to \hat{H}^i(G, H^k(C)) \to \hat{H}^{i+1}(G, \delta C^{k-1}) \to \hat{H}^i(G, \delta C^k) \to \ldots$$

Now, by the same argument as above, we get

$$\exp \hat{H}^{-1}(G, \delta C^0) \qquad \text{divides} \qquad \prod_{k=1}^{n} \hat{H}^{-k-2}(G, H^k(C))$$

From the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\epsilon} C^0 \to \delta C^0 \to 0$, we get

$$\hat{H}^{-1}(G, \delta C^0) \cong \hat{H}^0(G, \mathbb{Z}) \cong \mathbb{Z}/|G|$$

from which the second part of the theorem follows.

□

The equidimensional case of the main conjecture now follows easily.

Corollary 1. *Let $G = (\mathbb{Z}/p)^k$. If G acts freely on $X = \prod_{i=1}^n S^m$ and trivially on $H_*(X)$, then $k \leq n$.*

Proof. If G acts trivially on $H_*(X)$, then $\exp H^{i+1}(G, H_i(C)) = p$ for each i. It follows from Browder's theorem that $|G| = p^k | p^n$, which implies $k \leq n$.

□

This result requires in addition to the equidimensionality of the spheres that G act trivially on the homology of the space. This condition is not too restrictive, however; it is met for all subgroups of topological groups, for example. Adem and Browder have worked on the case where G does not act trivially on the homology, but this is out of the scope of this paper.

7 References

1. Browder, William: Cohomology and group actions
2. Brown, Kenneth: Cohomology of groups
3. Duan, Zhipeng: Group actions on a product of spheres, emphasizing on simple rank two groups
4. Okutan, Osman; Yalcin, Ergun: Free actions on products of spheres at high dimensions