Lie Algebra Cohomology and the Structure of Semisimple Lie Algebras

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The goal of this paper is to present an overview of the theory of Lie algebra cohomology and show some its ramifications. For the latter, we will prove two major theorems of Lie theory via cohomology of Lie algebras. In particular, we will establish two lemmas due to Whitehead regarding the cohomology groups of semisimple Lie algebras and use these to deduce the theorems due to Weyl and Levi-Mal'cev, which describe the sturcture of semisimple Lie algebra. In order to do this, we need to construct a cohomology theory for Lie algebras.

Just as Lie algebras give us a way to study Lie groups in a purely algebraic setting, Lie algebra cohomology gives us a purely algebraic way of studying the cohomology of the underlying topological space of a Lie group under the right conditions. This follows from a result due to Chevalley and Eilenberg, which says that the standard real cohomology of the underlying topological space of a compact connected Lie group is isomorphic to the real cohomology of its Lie algebra, to be defined below [1]. Thus, we will develop the general theory of Lie algebra cohomology algebraically and then study semisimple Lie algebras as a special case.

1 Theoretical Introduction

We start with some basic algebraic definitions. Let $\mathfrak{g}$ be a vector space over a field k. Define $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $[\ ,\ ]$ is bilinear and satisfies the following identities:

$$[x, x] = 0, \qquad x \in \mathfrak{g}$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \qquad x, y, z \in \mathfrak{g}$$

Then $\mathfrak{g}$ together with $[\ ,\ ]$ is called a *Lie algebra* with *Lie bracket* $[\ ,\ ]$. A *Lie algebra homomorphism* is then a linear map $f : \mathfrak{g} \to \mathfrak{h}$ which preserves Lie brackets, i.e. $f([x, y]) = [f(x), f(y)]$. A *Lie subalgebra* of $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ which is closed under the Lie bracket. A *Lie ideal* of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h}$ such that $[x, h] \in \mathfrak{h}$ for all $x \in \mathfrak{g}$ and $h \in \mathfrak{h}$. Lie ideals allow us to define quotient Lie algebras in the usual way. A Lie algebra (or a subalgebra or ideal) is called *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

1.1 Group Cohomology

Before defining Lie algebra cohomology, it is natural to define group cohomology, since the former construction is directly analogous to the latter. Recall that in order to define a cohomology theory for groups, we pass from a group G to the group ring $\mathbb{Z}G$ by imposing a natural ring structure on G. [2, Chapter 6]. This allows us to define G-modules as $\mathbb{Z}G$-modules. Then for any G-module A, we define the nth cohomology group of G with coefficients in A to be

$$H^n(G, A) = \mathrm{Ext}^n_G(\mathbb{Z}, A) = R^n(\mathrm{Hom}_G(\mathbb{Z}, A))$$

i.e. the nth right derived functor of the group of $\mathbb{Z}G$-module homomorphisms from $\mathbb{Z}$ to A. Here, $\mathbb{Z}$ denotes the trivial G-module.

Explictly, the construction of the homology groups proceeds as follows: Form a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$

$$P: \ \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

i.e. an exact sequence in which P_k is projective as a G-module for all k. Now, form the cochain complex

$$\mathrm{Hom}_G(P, A): \ 0 \to \mathrm{Hom}_G(P_0, A) \to \mathrm{Hom}_G(P_1, A) \to \dots$$

where we dualize P and remove the $\mathbb{Z}$ term. The nth homology of this complex is the $\mathrm{Ext}^n_G(\mathbb{Z}, A)$, i.e. the nth cohomology of G with coefficients in A.

In order to define cohomology of groups in this way, we needed to extend groups in a natural way to rings, over which we can define modules and chain complexes. We wish to make an analogous construction for Lie algebras. In this case, we will extend Lie algebras in a natural way to (associative) algebras, so we can define the notion of a $\mathfrak{g}$-module, where $\mathfrak{g}$ is a Lie group. First, note that the naturality of the construction $G \to \mathbb{Z}G$ is characterized by the following adjunction:

Theorem 1. *Let U : **Rng** $\to$ **Grp** be the functor which maps each ring (with unity) to its group of units. Then the functor $\mathbb{Z}(-)$: **Grp** $\to$ **Rng**, $G \mapsto \mathbb{Z}G$, which maps each group to its group ring, is a left adjoint of U.*

Proof. Let G be a group and R a ring with unity. Let $f : G \to U(R)$ be a group homomorphism. Define $f' : \mathbb{Z}G \to R$ by extending f linearly, i.e.

$$f'\left(\sum_i c_i x\right) = \sum_i c_i f(x)$$

for $c_i \in \mathbb{Z}$ (only finitely many nonzero) and $x \in G$. Multiplication in $\mathbb{Z}G$ is given by the operation in G, so f' preserves multiplication by assumption. Thus, f' is a ring homomorphism. Thus, we can extend each group homomorphism $f : G \to U(R)$ to a ring homomorphism $f' : \mathbb{Z}G \to R$. It is clear that we can also restrict ring homomorphisms $f : \mathbb{Z}(G) \to R$ to group homomorphisms

$f'' : G \to U(R)$, and that these processes are inverses, i.e. $f' \circ f'' = \mathrm{id}_{\mathbb{Z}G}$ and $f'' \circ f' = id_G$. Thus, we have a bijection

$$\mathrm{Hom}(\mathbb{Z}G, R) \cong \mathrm{Hom}(G, U(R))$$

Naturality follows from the fact that $\mathbb{Z}(f(G)) = f'(\mathbb{Z}(G))$ and $U(f(R)) = f''(U(R))$. This establishes the desired adjunction.

□

1.2 The Universal Enveloping Algebra

We now make a similar construction for a Lie algebra $\mathfrak{g}$. First, define a functor $L : \mathbf{Alg} \to \mathbf{LieAlg}$ by defining the Lie bracket $[\ ,\ ]$ on $L(A)$ in the classical way, i.e. $[x, y] = xy - yx$ for all $x, y \in A$. It is well-known that this is a Lie bracket. We now construct a left adjoint to U. To do this, let $\mathfrak{g}$ be a Lie algebra, let V be its underlying vector space, and define the *tensor algebra* TV to be

$$TV = \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

where $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ is the n-fold tensor product of V over k, with multiplication defined by

$$(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n) = x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n$$

It is clear with this definition that TV is a graded associative algebra. Consider the ideal $I \subset TV$ generated by elements of the form

$$x \otimes y - y \otimes x - [x, y], \qquad x, y \in \mathfrak{g}$$

Now form the quotient $U\mathfrak{g} = TV/I$. Note that the relation which generates I ensures that $[x, y] = x \otimes y - y \otimes x$ in $U\mathfrak{g}$ (where each term in this identity stands for its equivalence class in the quotient), which is made to be analogous to the classical Lie bracket defined above, $[x, y] = xy - yx$. We call $U\mathfrak{g}$ the *universal enveloping algebra* of $\mathfrak{g}$ and claim it is the desired left adjoint to L.

Theorem 2. $U(-) : \mathbf{LieAlg} \to \mathbf{Alg}$ *is left adjoint to* $L : \mathbf{Alg} \to \mathbf{LieAlg}$.

Proof. Let A be an algebra and $\mathfrak{g}$ a Lie algebra. Let $f : \mathfrak{g} \to L(A)$ be a Lie algebra homomorphism. We extend this to an algebra homomorphism $f' : T\mathfrak{g} \to A$ by letting $f'(x \otimes y) = f(x)f(y)$ for all $x, y \in \mathfrak{g}$. This defines $f'(x)$ for all $x \in T\mathfrak{g}$ by induction. Now, since f is a Lie algebra homomorphism, it preserves Lie brackets. Thus, $f'(x \otimes y - y \otimes x - [x, y]) = f(x)f(y) - f(y)f(x) - [f(x), f(y)] = 0$, since $[x, y] = xy - yx$ on $L(A)$. (We now see explicitly why I was defined in this way.) Hence, f' vanishes on I, and consequently induces an algebra homomorphism $f'' : U\mathfrak{g} \to A$. Uniquess and naturality follow in the same way as above. The result follows.

□

The important part of this theorem for our purposes is the fact that Lie algebra homomorphisms into $L(A)$ can be uniquely extended to algebra homomorphisms on the universal enveloping algebra. If $f : \mathfrak{g} \to L(A)$ is a Lie algebra homomorphism, let $f' : U\mathfrak{g} \to A$ denote this extension.

Now, let A be a vector space over k with a Lie algebra homomorphism $\phi : \mathfrak{g} \to L(\mathrm{End}(A))$. Using the observation above, we have a unique extension $\phi' : U\mathfrak{g} \to \mathrm{End}(A)$. This gives us a $U\mathfrak{g}$ action on A and makes A into a $U\mathfrak{g}$-module. We thus say that A is a $\mathfrak{g}$-module and call ϕ a $\mathfrak{g}$-action on A. The Lie algebra homomorphism condition ensures us that $\phi([x, y]) = [\phi(x), \phi(y)] = \phi(x)\phi(y) - \phi(y)\phi(x)$.

Note that in this case, $L(\mathrm{End}(A))$ is exactly the general linear Lie algebra of A, i.e. $\mathrm{gl}(A)$. Thus, $\mathfrak{g}$-actions on vector spaces are exactly Lie algebra representations. Note that from this we recover the usual definition of the adjoint representation of a Lie algebra, i.e.

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g}), \qquad x \mapsto (y \to [x, y])$$

This shows us that each Lie algebra is a module over itself with action given by its adjoint representation. This gives us a very natural and common example of a $\mathfrak{g}$-module.

Another simple but important example is that of a trivial $\mathfrak{g}$-module. This is any $\mathfrak{g}$-module with a trivial representation, i.e. $\phi(x)(a) = 0$ for all $x \in \mathfrak{g}$ and $a \in A$. This representation has a trivial Lie bracket, so a trivial $\mathfrak{g}$-module is just a vector space over k, and conversely, any vector space over k can be given a trivial Lie bracket and thus regarded as a trivial $\mathfrak{g}$-module for any Lie algebra $\mathfrak{g}$.

1.3 Lie Algebra Cohomology

We are now ready to define the cohomology of Lie algebras. We will adopt a similar convention to group cohomology and write $\mathfrak{g}$ in place of $U\mathfrak{g}$ when referring to Hom or Ext functors over $\mathfrak{g}$-modules. Using this convention, we define the nth cohomology group of $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module A by

$$H^n(\mathfrak{g}, A) = \mathrm{Ext}^n_{\mathfrak{g}}(k, A)$$

where k plays the role of $\mathbb{Z}$ in the theory of group cohomology, i.e. of a trivial $\mathfrak{g}$-module. Note that $H^n(\mathfrak{g}, A)$ has a natural $\mathfrak{g}$-module structure inherited from the $\mathfrak{g}$-module structure of k and its resolutions (so in particular, they are all vector spaces over k), but in the results that follow, we will only be concerned with the underlying group structure.

The computation of these cohomology groups from this definition is directly anlaogous to the procedure outlined above for groups. However, in the case of Lie algebras, there is a natural way of constructing projective resolutions that greatly simply the computation. This is given by the exterior algebra of the a vector space. We outline this construction below for general vector spaces.

Let V be a vector space over k. The exterior algebra of V will be a graded algebra related to the tensor algebra defined above. Start by considering the nth tensor power $V^{\otimes n}$ of V. Let I be the ideal of $V^{\otimes n}$ generated by vectors of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ where $x_i = x_j$ for some $i \neq j$. Let $V^{\wedge n} = V^{\otimes n}/I$. We call $V^{\wedge n}$ the *nth exterior power* of V.

We write $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for the image of $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ under the quotient map. This defines an alternating multilinear product (via the universal property of the tensor product) $\bigwedge : V^n \to V^{\wedge n}$ given by $(x_1, x_2, \ldots, x_n) \mapsto x_1 \wedge x_2 \wedge \cdots \wedge x_n$. Alternating in this case means that $x_i \wedge x_j = -x_j \wedge x_i$ for all $x_i, x_j \in V$, which is equivalent to $x_i \wedge x_i = 0$ for all $x_i \in V$, and generalizations of these properties to higher exterior powers follow by induction. Note also that $V^{\wedge 0} = k$ and $V^{\wedge 1} = V$. These follow immediately from the same relations on the tensor powers.

We now define the *exterior algebra* of V to be

$$\bigwedge^* V = \bigoplus_{n=0}^{\infty} V^{\wedge n}$$

It follows from the above properties that $\bigwedge^* V$ is an anti-commutative graded algebra.

We now return to Lie algebra cohomology. Let V be the underlying vector space of a Lie algebra $\mathfrak{g}$ over k. Now form the collection of $\mathfrak{g}$-modules $C_n = U\mathfrak{g} \otimes V^{\wedge n}$.

From this, we form the sequence

$$C: \ \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} k \to 0$$

where $\epsilon : C_0 = U\mathfrak{g} \to k$ is the unique homomorphism induced by the trivial representation $\mathfrak{g} \to \mathrm{gl}(k)$ (via the adjunction in theorem 2), and

$$\begin{aligned} d_n(x_1 \wedge \cdots \wedge x_n) = &\sum_{i=1}^{n} (-1)^{i+1} x_i \otimes x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_n \\ &+ \sum_{1 \le i < j \le n} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_n \end{aligned}$$

where a hat denotes that the element is removed from the product. Note that it is sufficient to define d_n on $V^{\wedge n}$ since this extends uniquely to $C_n = U\mathfrak{g} \otimes V^{\wedge n}$ by linearity. These maps are $\mathfrak{g}$-module homomorphisms by construction.

We can easily see that each C_n is a free $\mathfrak{g}$-module, i.e. a direct sum of $U\mathfrak{g}$. Explicitly, since $V^{\wedge n}$ is a vector space, we can choose a basis $\{v_i\}$ for $V^{\wedge n}$ and write

$$C_n = U\mathfrak{g} \otimes V^{\wedge n} = \bigoplus_i U\mathfrak{g} \otimes v_i$$

which is clearly isomorphic to a direct sum of $U\mathfrak{g}$, since $\mathfrak{g}$ acts trivially on $V^{\wedge n}$. Since free modules are projective, it follows that each C_n is a projective $\mathfrak{g}$-module.

Our goal is to use the sequence C to compute the cohomology groups $H^n(\mathfrak{g}, A)$. In order to do this, we need to show that C is a projective resolution of k over $\mathfrak{g}$. This requires showing that C is exact. The proof of this fact is long and computationally messy. Since we are more interested in the ramifications of the theory, we refer to [3, Lemma 4.5, Theorem 4.6].

In summary,

Proposition 1. *If $C_n = U\mathfrak{g} \otimes V^{\wedge n}$, where V is the underlying vector space of $\mathfrak{g}$, and $d_n : C_n \to C_{n-1}$ is given as above, then*

$$C: \ \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} k \to 0$$

is a projective (indeed free) resolution of k over $\mathfrak{g}$.

We can thus use this resolution to compute the cohomology groups in the same way as we did for groups, namely as the homology of the cochain complex given by

$$\mathrm{Hom}_{\mathfrak{g}}(P, A): \ 0 \to \mathrm{Hom}_{\mathfrak{g}}(C_0, A) \xrightarrow{d_1^*} \mathrm{Hom}_{\mathfrak{g}}(C_1, A) \xrightarrow{d_2^*} \mathrm{Hom}_{\mathfrak{g}}(C_2, A) \to \dots$$

where d_n^* is the induced coboundary map from d_n. Also, since elements of $\mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes V^{\wedge n}, A)$ correspond to alternating multilinear $\mathfrak{g}$-module homomorphisms $\mathfrak{g}^n \to A$, we simply write $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^n, A)$ for brevity.

2 Applications to the Structure of Lie Algebras

The importance of this particular construction of the cohomology groups is not in the proof that it works, but in the way it simplifies the computation of cohomology groups for specific Lie algebras by furnishing a concrete boundary map in terms of the vector space structure of the Lie algebra. To see some applications of this fact, we focus our attention on a special case. First, we take a brief detour to outline the essential aspects of the theory of semisimple Lie algebras.

2.1 Semisimplicity

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k of characteristic 0. For brevity, we refer to Lie ideals of $\mathfrak{g}$ simply as ideals. Let $\mathcal{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and define the collection of maps $\{\mathcal{D}^k\}$ inductively by $\mathcal{D}^n\mathfrak{g} = [\mathcal{D}^{n-1}\mathfrak{g}, \mathcal{D}^{n-1}\mathfrak{g}]$. it is clear that $\mathcal{D}^{n+1}\mathfrak{g} \leq \mathcal{D}^n\mathfrak{g}$ for all $n \geq 0$ (with the convention $\mathcal{D}^0\mathfrak{g} = \mathfrak{g}$). We call the sequence of Lie subalgebras $\{\mathcal{D}^k\mathfrak{g}\}$ the *derived series* of $\mathfrak{g}$. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *solvable* if its derived series terminates at 0, i.e. $\mathcal{D}^n\mathfrak{g} = 0$ for some n. In this case, we call n the *derived length* of $\mathfrak{g}$. By the second isomoprhism theorem, the set of solvable ideals in $\mathfrak{g}$ form a poset under containment in which all ascending chains meet. It follows that there is a unique largest solvable ideal in $\mathfrak{g}$. We call this the *radical* of $\mathfrak{g}$ and denote it $\operatorname{rad}\mathfrak{g}$. If $\operatorname{rad}\mathfrak{g} = 0$, we say that $\mathfrak{g}$ is *semisimple*, i.e. the only solvable ideal of $\mathfrak{g}$ is trivial.

Now, let A be a finite dimensional $\mathfrak{g}$-module with $\mathfrak{g}$-action given by $\rho : \mathfrak{g} \to \mathrm{gl}(A)$. Let $\beta : \mathfrak{g} \times \mathfrak{g} \to k$ be the bilinear form given by

$$\beta_\rho(x, y) = \mathrm{Tr}(\rho(x)\rho(y))$$

We note that this is a generalization of the Killing form: If we let $\mathfrak{g}$ be a $\mathfrak{g}$-module with action given by the adjoint representation, then we recover

$$\beta_{\mathrm{ad}}(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$$

This is called the *Killing form* of $\mathfrak{g}$. For shorthand, write $\langle x, y\rangle = \beta_{\mathrm{ad}}(x, y)$.

Recall the following important result, known as *Cartan's criterion for solvability*:

Proposition 2. *A Lie algebra $\mathfrak{g}$ is solvable if and only if $\langle x, [y, z]\rangle = 0$ for all $x, y, z \in \mathfrak{g}$.*

In particular, a Lie algebra is solvable if its Killing form is identically zero. This result can be found in *Lie Groups, Lie Algebras, and their Representation* by Varadarajan.

We now seek to prove the following important theorem. In what follows, we will assume that A is a finite dimensional $\mathfrak{g}$-module for a finite-dimensinal Lie algebra $\mathfrak{g}$ over a field k of characteristic 0.

Theorem 3. *Let A be a $\mathfrak{g}$-module with $\mathfrak{g}$-action given by $\rho : \mathfrak{g} \to \mathrm{gl}(A)$. Let $\mathfrak{h} = \ker\rho$. If $\mathfrak{g}$ is semisimple, then β_ρ is non-degenerate on $\mathfrak{h}^\perp \times \mathfrak{h}^\perp$.*

In this theorem, $\mathfrak{h}^\perp$ is the orthogonal complement of $\mathfrak{h}$ with respect to β_ρ. Before proving this theorem, we observe an important corollary:

Corollary 1. *If A is a $\mathfrak{g}$-module with faithful $\mathfrak{g}$-action ρ (i.e. ρ is injective), then β_ρ is non-degenerate on $\mathfrak{g} \times \mathfrak{g}$. .*

Proof. If ρ is injective, then $\mathfrak{h} = \ker\rho = 0$, hence $\mathfrak{h}^\perp = \mathfrak{g}$.

□

We now proceed to the proof of the theorem. First, we need a lemma:

Lemma 1. *Let $\mathfrak{g}$ be semisimple and $\mathfrak{h}$ an ideal in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$.*

Proof. First, note that, from elementary linear algebra, $\dim\mathfrak{g} = \dim\mathfrak{h} + \dim\mathfrak{h}^\perp$. Thus, the result follows as long as $\mathfrak{h}$ and $\mathfrak{h}^\perp$ have non-trivial intersection. Assume $\mathfrak{h} \cap \mathfrak{h}^\perp \neq 0$. Then $\langle x, x'\rangle = 0$ for all $x \in \mathfrak{h}$ and $x' \in \mathfrak{h}^\perp$. Thus, by Cartan's criterion for solvability, $\mathfrak{h} \cap \mathfrak{h}^\perp$ is solvable, contradicting the assumption that $\mathfrak{g}$ is semisimple.

□

This lemma allows us to form the split short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$$

That is, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$. It follows that each of $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are semisimple when $\mathfrak{g}$ is.

Now, recall the conditions of the theorem. $\mathfrak{h} = \ker \rho$ is an ideal of $\mathfrak{g}$ and we can thus write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. Let $\bar{\beta}_\rho : \mathfrak{h}^{\perp} \times \mathfrak{h}^{\perp} \to k$ denote the restriction of β_ρ to $\mathfrak{h}^{\perp} \times \mathfrak{h}^{\perp}$. Let $\mathfrak{m} = \ker \bar{\beta}_\rho(-, \mathfrak{h}^{\perp})$. $\mathfrak{m}$ is an ideal of $\mathfrak{g}$.

Choose any $x \in [\mathfrak{m}, \mathfrak{h}^{\perp}]$. Then we can see that $\rho(x)$ is nilpotent by considering $\mathrm{Tr}(\rho(x)r)$, where r is a replica of $\rho(x)$. [4, Lemma 3.9.5]. It follows that $p([\mathfrak{m}, \mathfrak{h}^{\perp}])$ is a nilpotent ideal of $\rho(\mathfrak{h}^{\perp})$. But the above shows that $\mathfrak{h}^{\perp}$ is seisimple, hence so is $\rho(\mathfrak{h}^{\perp})$ and $p([\mathfrak{m}, \mathfrak{h}^{\perp}])$. It follows that $p([\mathfrak{m}, \mathfrak{h}^{\perp}]) = 0$. Since ρ is injective on $\mathfrak{h}^{\perp}$, this implies $[\mathfrak{m}, \mathfrak{h}^{\perp}] = 0$. Thus, $\mathfrak{m} \leq Z(\mathfrak{h}^{\perp}) = 0$, since $\mathfrak{h}^{\perp}$ is semisimple, so $\mathfrak{m} = 0$ and the result follows.

2.2 First Calculations

We are now in a position to exploit the bilinear form β_ρ to prove substantial results about the cohomology of semisimple Lie algebras. We start with the following, which asserts the triviality of all cohomology groups for simple modules over semisimple Lie algebras.

Theorem 4. *Let A be a $\mathfrak{g}$-module with non-trivial $\mathfrak{g}$-action given by $\rho : \mathfrak{g} \to \mathrm{gl}(A)$. If A is simple and $\mathfrak{g}$ is semisimple, then $H^n(\mathfrak{g}, A) = 0$ for all n.*

Proof. Let $\mathfrak{h} = (\ker \rho)^{\perp}$. We assumed ρ is non-trivial, so $\mathfrak{h}$ is non-zero. The above theorem then gives us that β_ρ is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$. Thus, we can choose bases $\{e_i\}$ and $\{e'_j\}$ for $\mathfrak{h}$ over k such that $\beta_\rho(e_i, e'_j) = \delta_{ij}$.

For $\mathfrak{g}$-modues M and N, let $f_M : M \to M$ be the $\mathfrak{g}$-module homomorphism given by

$$x \mapsto \sum_i e_i e'_i x, \qquad x \in M$$

To show that f_M is a $\mathfrak{g}$-module, the only thing which is unclear is whether it preserves scalar multiplication in $U\mathfrak{g}$. Given the definition of the map, this amounts to proving that, for all $x \in U\mathfrak{g}$, x commutes with $\sum_i e_i e'_i$, i.e. $x \sum_i e_i e'_i = \sum_i e_i e'_i x$.

To show this, we first rewrite each side of the equation as follows:

$$x \sum_i e_i e'_i = \sum_i ([x, e_i]e'_i + e_i x e'_i)$$

$$\sum_i e_i e'_i x = \sum_i (e_i x e'_i - e_i[x, e'_i])$$

Now, expand each of the Lie brackets over the bases, i.e. $[e_i, x] = \sum_k c_{ik} e_k$ and $[x, e'_i] = \sum_k d_{ik} e_k$. Then, recalling that $\beta_\rho(e_i, e'_j) = \delta_{ij}$, we have

$$c_{ij} = \beta_\rho([e_i, x], e'_j) = \beta_\rho(e_i, [x, e'_j]) = d_{ji}$$

Expanding the Lie brackets over the bases in this way and equating c_{ij} with d_{ji}, we get

$$\sum_i ([x, e_i]e'_i + e_i x e'_i) = \sum_i (e_i x e'_i - e_i[x, e'_i])$$

which shows that $x \sum_i e_i e'_i = \sum_i e_i e'_i x$. Since x was arbitrary, it follows that f_M is indeed a $\mathfrak{g}$-module homomorphism.

Now, we make a similar construction for the modules $\{C_n\}$. Let C be the projective resolution furnished of k over $\mathfrak{g}$ furnished in proposition 1. Then $\tau : C \to C$ given by $\tau_n = f_{C_n}$ defines a chain map. Explicitly,

$$\tau_n : C_n \to C_n, \qquad x_1 \wedge x_2 \wedge \cdots \wedge x_n \mapsto \sum_i e_i e'_i \otimes x_1 \wedge x_2 \wedge \cdots \wedge x_n$$

It is clear from the above computation that τ_n is a $\mathfrak{g}$-module homomorphism for all n. The additional chain map condition says that $\tau_{n-1} d_n = d_n \tau_n$ for all n. We can deduce this from a more general property: For any $\mathfrak{g}$-module homomorphism $\phi : M \to N$, we have $f_N \phi = \phi f_M$. This follows immediately from the fact that each of the maps involved is a $\mathfrak{g}$-module homomorphism. It follows that τ is indeed a chain map.

Now, consider the map $f_A : A \to A$. The maps f_A and f_{C_n} each induce a map $\mathrm{Hom}(C_n, A) \to \mathrm{Hom}(C_n, A)$. We can write these explicitly as follows: If $\phi : C_n \to A$ is a $\mathfrak{g}$-module homomoprhism, then

$$f_A^* : \phi \mapsto f_A \phi$$

$$\tau_n^* : \phi \mapsto \phi \tau_n$$

It follows from the above observation that $f_A \phi = \phi \tau_n$, hence $f_A^* = \tau_n^*$ for all n. Furthermore, each of these maps induce a map on the cohomology groups $H^n(\mathfrak{g}, A) \to H^n(\mathfrak{g}, A)$. The fact that they are equal shows that they induce the same map; call it $f_* : H^n(\mathfrak{g}, A) \to H^n(\mathfrak{g}, A)$.

Now, A simple implies that f_A is either an automorphism of A or trivial. However, it is clearly not trivial by construction; hence it is an automorphism of A. It follows that f_* is an isomorphism. Thus, if we can show that f_* is trivial, it will follow that $H^n(\mathfrak{g}, A) = 0$ for all n, as claimed.

To do this, we construct a chain homomotopy between τ and the trivial map. Define a $\mathfrak{g}$-module homomoprhism $h_n : C_n \to C_{n+1}$ by

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n \mapsto \sum_k e_k \otimes e'_k \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_n$$

This is clearly a $\mathfrak{g}$-module homomorphism by construction. To show that h_n is a chain homotopy, we need $d_{n+1}h_n + h_{n-1}d_n = \tau_n$, where h_{-1} is defined to be trivial. We compute

$$
\begin{aligned}
(d_{n+1}h_n + h_{n-1}d_n)(x_1 \wedge \cdots \wedge x_n) = \sum_k e_k e_k' \otimes x_1 \wedge \cdots \wedge x_n \\
&+ \sum_{i,k} (-1)^i e_k \otimes [e_k', x_i] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_n \\
&+ \sum_{i,k} (-1)^i e_k x_i \otimes e_k' \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_n \\
&+ \sum_{i,k} (-1)^{i+1} x_i e_k \otimes e_k' \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_n \\
&+ \sum_{k,i<j} (-1)^{i+j} e_k \otimes [x_i, x_j] \wedge e_k' \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_n \\
&+ \sum_{k,i<j} (-1)^{i+j} e_k \otimes e_k' \wedge [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_n \\
&= \tau_n(x_1 \wedge \cdots \wedge x_n)
\end{aligned}
$$

In this massive sum of sums, every term except after the first sum cancels. The theorem follows.

□

2.3 Whitehead's Lemmas

We now proceed to the main results of this paper, namely the theorems due to Weyl and Levi-Malcev. Respectively, these theorems say that every finite dimensional module over a semisimple Lie algebra is a direct sum of simple modules, and that every finite-dimensional Lie algebra is the split extension of a semisimple Lie algebra by its radical. The first of these theorems implies that every finite-dimensional representation of a semisimple Lie algebra is completely reducible; the second allows you to decompose any finite-dimensional Lie algebra into a semisimple Lie subalgebra and a solvable Lie subalgebra (specifically, its radical). These theorems will follow as a result of two lemmas due to Whitehead, which state that the first and second cohomology groups of modules over semisimple Lie algebras are trivial. In these computations, we will reduce to the case of simple modules, which was proven above. Once again, all modules and Lie algebras will be finite-dimensional throughout, and all fields of characteristic 0.

Theorem 5. *Let A be a $\mathfrak{g}$-module with non-trivial $\mathfrak{g}$-action. If $\mathfrak{g}$ is semisimple, then $H^1(\mathfrak{g}, A) = 0$.*

Proof. Suppose the theorem is false. Then there exists a $\mathfrak{g}$-module A such that $H^1(\mathfrak{g}, A)$ is non-trivial. Let A be a $\mathfrak{g}$-module of minimal dimension with this property. If A is simple, then we are done by the previous theorem. Thus, assume A is not simple. Then A has a non-trivial proper submodule (ideal) A'. Form the canonical short exact sequence

$$0 \to A' \to A \to A/A' \to 0$$

By basic homological algebra, there is an associated long exact sequence of cohomology groups

$$\cdots \to H^1(\mathfrak{g}, A') \to H^1(\mathfrak{g}, A) \to H^1(\mathfrak{g}, A/A') \to H^2(\mathfrak{g}, A') \to \ldots$$

But we chose A to be the $\mathfrak{g}$-module of minimal dimension such that $H^1(\mathfrak{g}, A)$ was non-trivial, and $\dim A'$ and $\dim A/A'$ are both strictly less than $\dim A$, so $H^1(\mathfrak{g}, A') = H^1(\mathfrak{g}, A/A') = 0$. Thus, the above sequence contains

$$\cdots \to 0 \to H^1(\mathfrak{g}, A) \to 0 \to \ldots$$

It follows immediately from exactness that $H^1(\mathfrak{g}, A) = 0$.

□

Theorem 6. *Let A be a $\mathfrak{g}$-module with non-trivial $\mathfrak{g}$-action. If $\mathfrak{g}$ is semisimple, then $H^2(\mathfrak{g}, A) = 0$.*

Proof. The proof will proceed as the previous one. Suppose the theorem is false. Let A be the $\mathfrak{g}$-module of minimal dimension such that $H^2(\mathfrak{g}, A) \neq 0$. If A is simple, then we are done by theorem 4. Thus, assume A is not simple. Let A' be a non-trivial proper submodule of A and form the canonical short exact sequence

$$0 \to A' \to A \to A/A' \to 0$$

and the associated long exact sequence

$$\cdots \to H^2(\mathfrak{g}, A') \to H^2(\mathfrak{g}, A) \to H^2(\mathfrak{g}, A/A') \to H^3(\mathfrak{g}, A') \to \ldots$$

In the same way as above, we get $H^2(\mathfrak{g}, A') = H^2(\mathfrak{g}, A/A') = 0$, which implies $H^2(\mathfrak{g}, A) = 0$.

□

Thus, we have shown that the first and second cohomology groups of a semisimple Lie algebra are trivial over non-simple modules and simple modules with non-trivial $\mathfrak{g}$-action. If A is simple and has trivial $\mathfrak{g}$-action, then $A = k$, the underlying field. Thus, in order to fully generalize these results to semisimple Lie algebras, we have only to show that $H^1(\mathfrak{g}, k) = H^2(\mathfrak{g}, k) = 0$ for any semisimple Lie algebra $\mathfrak{g}$. We compute these directly from the cochain complex

$$\mathrm{Hom}_{\mathfrak{g}}(P, A): \; 0 \to \mathrm{Hom}_{\mathfrak{g}}(C_0, A) \xrightarrow{d_1^*} \mathrm{Hom}_{\mathfrak{g}}(C_1, A) \xrightarrow{d_2^*} \mathrm{Hom}_{\mathfrak{g}}(C_2, A) \to \dots$$

where $C_n = U\mathfrak{g} \otimes V^{\wedge n}$ (recall proposition 1). Since we are only concerned with the first and second cohomology groups, we investigate the sequence

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g}, k) \xrightarrow{d_1^*} \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, k) \xrightarrow{d_2^*} \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^2, k) \xrightarrow{d_3^*} \dots$$

Now, k as a $\mathfrak{g}$-module has trivial $\mathfrak{g}$-action, i.e. $\rho(x)(a) = 0$ for all $x \in \mathfrak{g}$ and $a \in k$. Now, by dualizing the definition of d_1 above, we get

$$d_1^*\theta(a \otimes x) = \rho(x)\theta(a), \qquad \theta \in \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g}, k)$$

so the condition that ρ is the trivial action is equivalent to the assertion that $\mathrm{im}\, d_1^* = 0$. Thus, $H^1(\mathfrak{g}, k) = \ker d_2^*$. Similarly, we compute

$$d_2^*\theta(x_1 \wedge x_2) = \rho(x_1)\theta(x_2) - \rho(x_2)\theta(x_1) - \theta([x_1, x_2]) = -\theta([x_1, x_2])$$

for $\theta \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, k)$. Note that the first two terms are zero because the action is trivial.

The kernel of d_2^* is therefore the subspace of maps $\theta \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, k)$ satisfying $\theta([x, y]) = \rho(x_1)\theta(x_2) - \rho(x_2)\theta(x_1) = 0$. These are exactly the Lie algebra homomorphisms $\theta : \mathfrak{g} \to k$ with trivial image; in other words, the Lie algebra homomorphisms $\theta : \mathfrak{g}_{ab} \to k$, where $\mathfrak{g}_{ab} = \mathfrak{g}/[g, g]$ is the abelianization of $\mathfrak{g}$. Thus, we conclude that $H^1(\mathfrak{g}, k) = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}_{ab}, k)$. But $\mathfrak{g}_{ab}$ is an abelian ideal of $\mathfrak{g}$, hence solvable. Thus, since $\mathfrak{g}$ is semisimple, $\mathfrak{g}_{ab} = 0$, and we get $H^1(\mathfrak{g}, k) = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}_{ab}, k) = 0$, as desired.

Now, we compute $H^2(\mathfrak{g}, k) = \ker d_3^* / \mathrm{im}\, d_2^*$. The computation above shows that $\mathrm{im}\, d_2^*$ is the subspace of maps in $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^2, k)$ generated by $\theta([x, y])$ for $\theta \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, k)$ and $x, y \in \mathfrak{g}$. Now, to find $\ker d_3^*$, we compute

$$\begin{aligned} d_3^*\theta(x_1 \wedge x_2 \wedge x_3) &= \rho(x_1)\theta(x_2 \wedge x_3) - \rho(x_2)\theta(x_1 \wedge x_3) + \rho(x_3)\theta(x_1 \wedge x_2) \\ &\quad - \theta([x_1, x_2] \wedge x_3) + \theta([x_1, x_3] \wedge x_2) - \theta([x_2, x_3] \wedge x_1) \\ &= \theta([x_1, x_3] \wedge x_2) + \theta([x_2, x_1] \wedge x_3) + \theta([x_3, x_2] \wedge x_1) \end{aligned}$$

for $\theta \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^2, k)$. Once again, the first three terms are zero because the action is trivial. Thus, we see that the kernel of d_3^* is the subspace of maps $\theta \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^2, k)$ satisfying

$$\theta([x_1, x_3] \wedge x_2) + \theta([x_2, x_1] \wedge x_3) + \theta([x_3, x_2] \wedge x_1) = 0$$

Note that if our map is $\theta([x, y])$, then this is exactly the Jacobi identity, hence in particular it is satisfied by $\theta([x, y])$. Conversely, any map satisfying this identity must be a linear combination of maps of the form $\theta([x, y])$. Therefore, $\ker d_3^* = \mathrm{im}\, d_2^*$, and it follows that $H^2(\mathfrak{g}, k) = 0$, as desired.

We have therefore generalized the previous two theorems to the following:

Proposition 3. *Let A be a $\mathfrak{g}$-module. If $\mathfrak{g}$ is semisimple, then $H^1(\mathfrak{g}, A) = H^2(\mathfrak{g}, A) = 0$.*

The assertions that the first and second cohomology groups are trivial under these conditions are known as the Whitehead lemmas.

3 Main Structure Theorems

We now apply the Whitehead lemmas to prove two important structure theorems about semisimple Lie algebras due to Weyl and Levi-Malcev. First, we state Weyl's theorem.

Theorem 7. *Let A be a $\mathfrak{g}$-module. If $\mathfrak{g}$ is semisimple, then A is a direct sum of simple $\mathfrak{g}$-modules.*

We reiterate that an equivalent statement in terms of representation theory is that (finite-dimensional) representations of a semisimple Lie algebra are completely reducible.

Proof. If $\dim A = 0$, then there is nothing to prove. Assume we have the result for $\dim A \leq n$. Let A be a $\mathfrak{g}$-module of dimension $n+1$. Let A' be a non-trivial proper submodule of A. Form the canonical short exact sequence

$$0 \to A' \to A \to A/A'$$

and the induced exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(A/A', A') \to \mathrm{Hom}_{\mathfrak{g}}(A, A') \to \mathrm{Hom}_{\mathfrak{g}}(A', A') \to 0$$

(Note that the Hom functor is exact over fields.) Regarding each term in this sequence as a $\mathfrak{g}$-module via the action $(x\phi(a)) = x\phi(a) - \phi(xa)$, $x \in \mathfrak{g}$, $a \in A, A'$, or A/A', we can form the long exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H^0(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A/A', A')) \to H^0(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A, A')) \\ &\to H^0(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A', A')) \to H^1(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A'', A')) \to \dots \end{aligned}$$

We assumed that $\mathfrak{g}$ is semisimple, so Whitehead's first lemma shows that $H^1(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A'', A')) = 0$. Thus, we get a short exact sequence

$$0 \to H^0(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A/A', A')) \to H^0(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A, A')) \to H^0(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A', A')) \to 0$$

In particular, the map $H^0(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A, A')) \to H^0(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(A', A'))$ is surjective. This induces a surjective homomorphism $\mathrm{Hom}_{\mathfrak{g}}(A, A') \to \mathrm{Hom}_{\mathfrak{g}}(A', A')$. It follows that there is a $\mathfrak{g}$-module homomorphism $\phi : A \to A'$ which is the identity on A'. In other words, ϕ is the left inverse of the inclusion $i : A' \to A$

(the canonical map in the original short exact sequence). Thus, ϕ induces a splitting of the sequence, from which it follows that $A = A' \oplus A/A'$.

Since A' is a proper submodule of A, $\dim A'$ and $\dim A/A'$ are each strictly less than $\dim A$. By our assumption, A' and A/A' each decompose into the direct sum of simple $\mathfrak{g}$-modules. Therefore, the same is true of A. The result follows by induction.

□

Before moving onto the second major theorem, we prove a lemma.

Lemma 2. *Let A be a $\mathfrak{h}$-module and let $\mathfrak{g}$ be an extension of $\mathfrak{h}$ by A, i.e. a short exact sequence $0 \to A \to \mathfrak{g} \to \mathfrak{h} \to 0$ with abelian kernel. Then there is a unique element of $H^2(\mathfrak{h}, A)$ corresponding to $\mathfrak{g}$.*

Note that this is a special case of a bijective correspondence between Lie algebra extensions $0 \to A \to \mathfrak{g} \to \mathfrak{h}$ and $H^2(\mathfrak{h}, A)$, but we only need one direction of this correspondence for our purposes.

Proof. Let ρ be the $\mathfrak{h}$-action on A, as usual. Let $s : \mathfrak{h} \to \mathfrak{g}$ be a section, i.e. a linear map which is a right inverse of the projection $\mathfrak{g} \to \mathfrak{h}$. Define $h : \mathfrak{h} \times \mathfrak{h} \to A$ by

$$h(x, y) = [sx, sy] - s[x, y], \qquad x, y \in \mathfrak{h}$$

It is clear that h is alternating and multilinear (since the Lie bracket has each of these properties). Thus, we can regard h as a function in $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{g}^2, A)$. Without loss of generality, we can let h define the Lie bracket on $\mathfrak{g}$, since the section $s : \mathfrak{h} \to \mathfrak{g}$ describes $\mathfrak{g}$ as the semi-direct product of A and $\mathfrak{h}$ by h. That is, $\rho(x)(y) = h(x, y)$. Now, we compute

$$\begin{aligned}
d_3^* h(x_1 \wedge x_2 \wedge x_3) &= \rho(x_1)h(x_2 \wedge x_3) - \rho(x_2)h(x_1 \wedge x_3) + \rho(x_3)h(x_1 \wedge x_2) \\
&\quad - h([x_1, x_2] \wedge x_3) + h([x_1, x_3] \wedge x_2) - h([x_2, x_3] \wedge x_1) \\
&= [sx_1, [sx_2, sx_3]] + [sx_2, [sx_3, sx_1]] + [sx_3, [sx_1, sx_2]] \\
&\quad - s[x_1, [x_2, x_3]] + s[x_2, [x_3, x_1]] + s[x_3, [x_1, x_2]] = 0
\end{aligned}$$

It is clear that the last expression equals zero by linearity. Thus, $h \in \ker d_3^*$. It follows that h is an element of a cohomology class in $H^2(\mathfrak{h}, A)$. Now, suppose we followed the same procedure for a different section $r : \mathfrak{h} \to \mathfrak{g}$. Let $k : \mathfrak{h} \times \mathfrak{h} \to A$ be the associated 2-cochain cycle. Then

$$\begin{aligned}
d_2^*(r - s)(x_1 \wedge x_2 \wedge x_3) &= \rho(x_1)(r - s)(x_2) - \rho(x_2)(r - s)(x_1) - (r - s)([x_1, x_2]) \\
&= [rx_1, rx_2] - r[x_1, x_2] + [sx_1, sx_2] - s[x_1, x_2] \\
&= k(x_1 \wedge x_2) - h(x_1 \wedge x_2)
\end{aligned}$$

From this, it follows that sections with the same image under d_2^* give rise to the same 2-cochain cycles. Therefore, to each such extension there corresponds a unique element of $H^2(\mathfrak{h}, A)$.

$\square$

We now state the theorem due to Levi-Malcev.

Theorem 8. *Every finite-dimensional Lie algebra admits the following decomposition:* $\mathfrak{g} = \operatorname{rad}\mathfrak{g} \oplus \mathfrak{g}/\operatorname{rad}\mathfrak{g}$

Proof. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with radical $\operatorname{rad}\mathfrak{g}$. Form the canonical short exact sequence

$$0 \to \operatorname{rad}\mathfrak{g} \to \mathfrak{g} \to \mathfrak{g}/\operatorname{rad}\mathfrak{g}$$

It suffices to show that this sequence splits. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{rad}\mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{g}/\operatorname{rad}\mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{rad}\mathfrak{g}/[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}] \arrow[r] & \mathfrak{g}/[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}] \arrow[r] & \mathfrak{g}/\operatorname{rad}\mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

The bottom row is clearly exact as well. Note that the first term is the abelianization of $\operatorname{rad}\mathfrak{g}$. Thus, the bottom row gives an extension of $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ by $\operatorname{rad}\mathfrak{g}/[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$. It follows from the lemma that there is a unique element of $H^2(\mathfrak{g}/\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}/[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}])$ to which it corresponds. But this group is trivial by the second Whitehead lemma. Therefore, there must be only one such extension. Since the split extension always exists, this sequence must split.

Assume $\operatorname{rad}\mathfrak{g}$ has derived length 1. Then $[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}] = 0$, hence $\operatorname{rad}\mathfrak{g}$ is abelian, and so the top row splits as well by the above. Now assume the result is known for all Lie algebras with radical of derived length $\leq n$. Consider the case where $\operatorname{rad}\mathfrak{g}$ has derived length $n + 1$.

Let $s : \mathfrak{g}/\operatorname{rad}\mathfrak{g} \to \mathfrak{g}/[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$ be a splitting. Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ such that $\operatorname{im} s = \mathfrak{h}/[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$. Form the canonical short exact sequence

$$0 \to [\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}] \to \mathfrak{h} \to \mathfrak{h}/[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}] \to 0$$

Since $\operatorname{rad}\mathfrak{g}$ has derived length $n+1$, $[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$ has derived length n, hence this sequence splits by the assumption. Let $r : \mathfrak{h}/[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$ be a splitting. Now define $q : \mathfrak{g}/[\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}] \to \mathfrak{g}$ by $q = rs$. It is clear from construction that q gives a splitting of the top row. The result follows by induction.

$\square$

Thus, we have proved two major theorems regarding the structure of semisimple Lie algebras via cohomology, specifically the triviality of the first and second cohomology groups of modules over semisimple Lie algebras. It is natural to ask whether higher cohomology groups are also trivial and what results we may be able to deduce from that. The answer to this question is negative, as demonstrated by the following result due to Chevalley and Eilenberg. [1, Theorem 21.1].

Theorem 9. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $H^3(\mathfrak{g}, k) \neq 0$.*

Proof. Recall the Killing form $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to k$ given by $\langle x, y\rangle = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$. Define an alternating multilinear functional $f : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to k$ by

$$f(x, y, z) = \langle [x, y], z\rangle$$

By construction (via the properties of the Lie bracket and Killing form), f is a 3-cochain cycle. Now, for $x \in \mathfrak{g}$, we introduce the auxiliary function $\omega_x : \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^3, k) \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^3, k)$ given by

$$\omega_x \phi(x_1 \wedge \cdots \wedge x_n) = \sum_i (-1)^{i+1} \phi([x_i, x] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_n)$$

Note that $d_k * \omega_x = \omega_x d_k^*$ for all k. We also have

$$\begin{aligned}\omega_x f(x_1 \wedge x_2 \wedge x_3) &= \langle [[x_1, x], x_2], x_3\rangle - \langle [[x_2, x], x_1], x_3\rangle + \langle [[x_3, x], x_1], x_2\rangle \\ &= 0\end{aligned}$$

i.e. $\omega_x f = 0$ for all $x \in \mathfrak{g}$. We use this fact to compute

$$\begin{aligned}d_4^* f(x_1 \wedge x_2 \wedge x_3 \wedge x_4) &= \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge x_{i'} \wedge x_{j'}) \\ &= \sum_i \omega_{x_i} f(x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge x_3) = 0\end{aligned}$$

Thus, f is a 3-cocycle. Assume f is a 3-cobondary. Then there is some $g \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^2, k)$ such that $d_3^* g = f$. The above observations then give us

$$d_3^* \omega_x g = \omega_x d_3^* g = \omega_x f = 0$$

Thus, $\omega_x g$ is a 2-cocycle, by a similar calcluation to the one above. It follows that g is also a 2-cocycle. Hence, $d_3^* g = f = 0$. But f cannot be trivial, because the Killing form is non-degenerate by the corollary to theorem 3 (since ad is a faithful representation). Thus, we have a contradiction. Therefore, f is a 3-cocycle that is not a 3-coboundary. It follows that the cohomology class of f is nontrivial, so $H^3(\mathfrak{g}, k) \neq 0$.

□

4 Connection Between Lie Algebra Cohomology and de Rahm Cohomology

The reader who is familiar with de Rahm cohomology probably observed some similarity between it and the definition of the Lie algebra cohomology earlier in this paper. In this section, we make this connection formal. In particular, we will prove that the de Rahm cohomology of a compact connected Lie group is isomorphic to the Lie algebra cohomology of its Lie algebra.

We will start by introducing Lie algebras from a different perspective. We let G denote a topological group which is also a smooth manifold and for which the group operation and inversion map are smooth. Then G is called a *Lie group*. Let $T_e(G)$ denote the tangent space of G at the identity, and let $L_r : G \to G$ denote the left translation $x \mapsto rx$. L_r induces a map $(L_r)_* : T_e(G) \to T_e(G)$, which we call the *differential* or *pushforward*. Thus, if we fix $X_e \in T_e(G)$ and consider the set of all $X_r = (L_r)_*(X_e)$, this forms a left-invariant vector field X on G which is uniquely determined by X_e. We can furthermore show that any such vector field is smooth.

More generally, for any p-form ω on G, L_r induces a map on the space of p-forms, namely $(L_r)^*(\omega)(X_1, \ldots, X_p) = \omega((L_r)_*(X_1), \ldots, (L_r)_*(X_p))$, which we call the *pullback*. We then call a p-form ω *left invariant* if it is fixed under the map $(L_r)^*$, i.e. $(L_r)^*(\omega) = \omega$. It can be shown that the left-invariant p-forms are in one-to-one correspondence with the p-forms on $T_e(G)$.

Next, observe that if we define the Lie bracket in the usual way, it preserves left-invariance, i.e. $[X, Y] = XY - YX$ is left-invariant whenever X and Y are. Thus, the Lie bracket is well defined on $T_e(G)$. Thus, $T_e(G)$ together with the Lie bracket $[\ ,\ ]$ defines a Lie algebra. We call this the Lie algebra associated to the Lie group G and denote it by $\mathfrak{g}$.

With this definition of the Lie algebra, we will need to modify the notation for the differential map between forms slightly, but it will be essentially the same as given above. We have, for a p-form ω,

$$\begin{aligned} d\omega(X_0, \ldots, X_p) = &\sum_{i=0}^{p} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)) \\ &+ \sum_{1 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) \end{aligned}$$

However, if ω and $X_0, \ldots, X_p$ are left-invariant, then $\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)$ is constant, hence $X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)) = 0$, and the above formula reduces to

$$d\omega(X_0, \ldots, X_p) = \sum_{1 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)$$

Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g}$ and $\omega_1, \ldots, \omega_n$ be the 1-forms dual to this basis, i.e. $\omega_i(X_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Let $\omega = \omega_1 \wedge \cdots \wedge \omega_n$. We see that ω is a left-invariant n-form. Now, for $f : G \to \mathbb{R}$ smooth, let

$$\int_G f dg = \int_G f\omega$$

This is called the *Haar integral.* It inherets the property of left-invariance from ω. Namely,

$$\int_G f(r\sigma)dg = \int_G f \circ L_r dg = \int_G f \circ L_r \omega = \int_G f\omega = \int_G f(\sigma)dg$$

Furthermore, if G is compact, we can show that the Haar integral is right-invariant as well. Now let M be a smooth manifold and define a smooth action $G \times M \to M$ of G on M by $(g, x) \mapsto t_g(x)$. A p-form ω is said to be invariant tif $t_g^*(\omega) = \omega$ for all $g \in G$. Let $\Omega(M)$ denote the set of forms on M, and $\Omega^G(M)$ denote the set of invariant forms on M with respect to the action of G. We now define a map $I : \Omega(M) \to \Omega^G(M)$ by

$$I(\omega)(X_1, \ldots, X_p) = \int t_g^*\omega(X_1, \ldots, X_p)dg = \int \omega((t_g)_*X_1, \ldots, (t_g)_*X_p)dg$$

With this definition, by the right-invariance of the Haar integral, we compute

$$\begin{aligned} t_h^*(I(\omega))(X_1, \ldots, X_p) &= I(\omega)((t_h)_*X_1, \ldots, (t_h)_*X_p) \\ &= \int \omega((t_g)_*(t_h)_*X_1, \ldots, (t_g)_*(t_h)_*X_p)dg \\ &= \int \omega((t_{gh})_*X_1, \ldots, (t_{gh})_*X_p)dg \\ &= \int \omega((t_g)_*X_1, \ldots, (t_g)_*X_p)dg \\ &= I(\omega) \end{aligned}$$

This shows that $I(\omega)$ is itself a left-invariant form on M for all $\omega \in \Omega^G(M)$. On the other hand, assume G is compact. Then we can normalize the Haar integral so that

$$\int_G dg = 1$$

Now, for $\omega \in \Omega^G(M)$, we use left-invariance to compute

$$
\begin{aligned}
I(\omega)(X_1,\ldots,X_p) &= \int t_g^*(\omega)(X_1,\ldots,X_p)dg \\
&= \int \omega(X_1,\ldots,X_p)dg \\
&= \omega(X_1,\ldots,X_p)\int_G dg \\
&= \omega(X_1,\ldots,X_p)
\end{aligned}
$$

Note that G acts by automorphisms on $H^p(M,\mathbb{R})$. Let $H^p(M,\mathbb{R})^G$ denote the set of fixed points under this action.

Now let $J : \Omega^G(M) \hookrightarrow \Omega(M)$ be the natural inclusion. The above computation shows that integrating left-invariant forms via I results in left-invariant forms. In other words, the composition $IJ : \Omega^G(M) \to \Omega^G(M)$ is the identity. Note that I and J induce homomorphisms in cohomology, namely

$$
I^* : H^p(M,\mathbb{R})^G \to H^p(\Omega^G(M)) \quad \text{and} \quad J^* : H^p(\Omega^G(M)) \to H^p(M,\mathbb{R})^G
$$

Since $IJ = 1$, we have $I^*J^* = 1$ as well. Therefore, I^* is surjective and J^* is injective. Thus, if we show that J^* is surjective, then we will have shown the following:

Theorem 10. *Let G be a compact Lie group. Then the inclusion $J : \Omega^G(M) \hookrightarrow \Omega(M)$ induces an isomprhism $J^* : H^p(\Omega^G(M)) \cong H^p(M,\mathbb{R})^G$.*

Proof. As shown just above, it suffices to how that J^* is onto. Choose $\alpha = [[\omega]] \in H^*(M,\mathbb{R})^G$. Let $g \in G$. Sincne $[[\omega]]$ is left-invariant under G, ω and $t_g^*(\omega)$ are in the same cohomology class. In other words, $\omega - t_g^*(\omega) = d\eta$ for some $(p-1)$-form η, depending on g. Therefore, for any smooth p-cycle $c \in C_p(M)$, we have, applying Stoke's theorem,

$$
\int_c \omega - \int_c t_g^*(\omega) = \int_c d\eta = \int_{\partial c} \eta = 0
$$

It follows that

$$
\begin{aligned}
\int_c I(\omega) &= \int_c \int_G t_g^*(\omega)dg \\
&= \int_G \int_c t_g^*(\omega)dg \\
&= \int_G \int_c \omega dg \\
&= \int_c \omega \int_G dg \\
&= \int_c \omega
\end{aligned}
$$

using the compactness of G to normalize the integral as above. Therefore, for every p-cycle c, we have

$$\int_c (I(\omega) - \omega) = 0$$

It follows that $[[I(\omega)]] = [[\omega]]$ in $H^*(M, \mathbb{R})$. Therefore, $J^* I *^(\alpha) = \alpha$ and the result follows.

□

This theorem gives us the immediate corollary:

Corollary 2. *Let G be a compact connected Lie group. Then $H^p(G, \mathbb{R})$ is isomorphic to $H^p(\mathfrak{g}, \mathbb{R})$.*

This follows from the following two observations: First, as stated above, $\mathfrak{g}$ is exactly $T_e(G)$ together with the Lie bracket, hence $H^p(\mathfrak{g}, \mathbb{R})$ is exactly the cohomology of the complex of invariant forms with differential given above. Second, when G is connected, $H^p(M, \mathbb{R})^G = H^p(G, \mathbb{R})$, since then t_g is the identity on M for each g in G.

5 References

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